$t \geq 0$ and $\pi_n = \{t_0 = 0, t_1, \dots, t_{n(\pi)} = t\}$ be a partition of $[0, t]$. We want to show that

$$\lim_{n(\pi)\to\infty} \sum_{k=1}^{n(\pi)} (W_{t_k} - W_{t_{k-1}})^2 =: V_n^{(2)}(W) = t$$

where the convergence is in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, that is,

$$\lim_{n(\pi)\to\infty} \mathbb{E}\left[\left(\sum_{k=1}^{n(\pi)} (W_{t_k} - W_{t_{k-1}})^2 - t\right)^2\right] = 0.$$

For brevity of notation, write $\Delta t_k = t_k - t_{k-1}$, $\Delta W_k = W_{t_k} - W_{t_{k-1}}$ and $\theta_k = (\Delta W_k)^2 - \Delta t_k$. Our goal is to prove that $I_n = \mathbb{E}[(\sum \theta_k)^2] \to 0$ as $n(\pi) \to \infty$. To show this,

$$I_n = \mathbb{E}\left[\left(\sum_{k=1}^{n(\pi)} \theta_k\right)^2\right] = \mathbb{E}\left[\sum_{i,j} \theta_i \theta_j\right] = \sum_{i=1}^{n(\pi)} \mathbb{E}(\theta_i^2) + 2\sum_{i\neq j} \mathbb{E}(\theta_i \theta_j)$$

We can show that the second term is 0 by using independent increments and $\mathbb{E}[(\Delta W_k)^2] = \Delta t_k$. Then

$$\begin{aligned} I_n &= \sum_{k=1}^{n(\pi)} \left(\mathbb{E}[(\Delta W_k)^4] - 2\Delta t_k \mathbb{E}[(\Delta W_k)^2] + (\Delta t_k)^2\right) \\ &= 2\sum_{k=1}^{n(\pi)} (\Delta t_k)^2 \leq 2|\pi_n| \sum_{k=1}^{n(\pi)} \Delta t_k = 2t|\pi_n| \end{aligned}$$

where in the second equality we used the fact the 4th central moment of a $X \sim \mathcal{N}(\mu, \sigma^2)$ r.v. is $\mathbb{E}((X-\mu)^4) = 3\sigma^4$.

It is now clear that the quantity I_n tends to zero as n tends to infinity, since $|\pi_n|$ tends to zero.

Lemma 0.0.1. *Th*