Proof. Let η_T be the Radon–Nikodym density of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ (whose existence is guaranteed by Radon–Nikodym theorem). Clearly, the process $\eta_t = \mathbb{E}^{\mathbb{P}}(\eta_T \mid \mathcal{F}_t)$ is an $\mathcal{F}$-martingale and $\eta_0 = 1$. Further, since the underlying filtration $\mathcal{F} = \mathcal{F}^W$ is a Brownian filtration, the martingale representation theorem (Theorem 5.4.2) implies the existence of a process $\widetilde{\gamma} \in \mathcal{L}_{\mathbb{P}}(W)$ such that

$$\eta_t = 1 + \int_0^t \widetilde{\gamma}_u \cdot d\boldsymbol{W}_u, \quad \forall t \in [0, T].$$

Since $\mathbb{P}\{\eta_T > 0\} = 1$, we also have that $\mathbb{P}\{\eta_t > 0\} = 1$ for any $t \in [0, T]$, and further, in view of continuity of η (which is apparent from the representation above) we obtain $\mathbb{P}\{\eta_t > 0, \forall t \in [0, T]\} = 1$. Therefore, the process $\gamma_t = \widetilde{\gamma}_t \eta_t^{-1}$ is well defined and we have

$$\eta_t = 1 + \int_0^t \widetilde{\gamma}_u \cdot d\boldsymbol{W}_u = 1 + \int_0^t \eta_u \gamma_u \cdot d\boldsymbol{W}_u.$$

We conclude that the Radon–Nikodym density process is the unique solution to the SDE

$$d\eta_t = \eta_t \gamma_t \cdot d\boldsymbol{W}_t,$$

and thus, in view of the form of the stochastic exponential, it satisfies, for any $t \in [0, T]$,

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \gamma_u \cdot d\boldsymbol{W}_u\right)$$

This completes the proof of the proposition. ■