Theorem 0.0.1 (conditional expectation)**.** *Let X, Y be two integrable random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and $\mathcal{G}$, $\mathcal{H}$ be two sub σ-algebras of $\mathcal{A}$. Then*

1. ***(taking out what is known)*** *If X is $\mathcal{G}$-measurable (or equivalently $\sigma(X) \subseteq \mathcal{G}$), then $\mathbb{E}[X|\mathcal{G}] = X$.*

2. ***(linearity)*** *For $a, b \in \mathbb{R}$, $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.*

3. ***(tower property)*** *If $\mathcal{H} \subseteq \mathcal{G}$ then*

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$$

in particular, by taking $\mathcal{H} = \{\emptyset, \Omega\}$ to be the trivial σ-algebra, we have

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}(X).$$

4. *If X is independent of $\mathcal{G}$ in the sense that for all $A \in \sigma(X)$ and $B \in \mathcal{G}$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then*

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X].$$

5. *If X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$ then for any Borel function $h : \mathbb{R}^2 \to \mathbb{R}$ we have*

$$\mathbb{E}[h(X,Y)|\mathcal{G}] = H(X),$$

where $H : \mathbb{R} \to \mathbb{R}$ is given by the formula $H(x) = \mathbb{E}[h(x, Y)]$. Consider, e.g. X represents the present, Y the future and $\mathcal{G}$ the information generated by past events.

6. ***(conditional Jensen's inequality)*** *Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function and for any σ-algebra $\mathcal{G} \subseteq \mathcal{A}$,*

$$g(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[g(X)|\mathcal{G}].$$