Definition 0.0.1 (conditional expectation w.r.t. a σ-algebra)**.** Let X be a integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. Given a arbitrary sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{A}$, the **conditional expectation** of X with respect to $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, is the unique integrable random variable satisfying the following conditions:

(i) (measurability) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable, i.e. $\mathbb{E}[X|\mathcal{G}]^{-1}(\mathcal{B}(\mathbb{R})) \subseteq \mathcal{G}$ or $\mathbb{E}[X|\mathcal{G}]^{-1}(B) \in \mathcal{G}$ for all $A \in \mathcal{B}(\mathbb{R})$.

(ii) (partial averaging) For any $A \in \mathcal{G}$, we have

$$\mathbb{E}[\mathbf{1}_A X] = \mathbb{E}[\mathbf{1}_A \mathbb{E}[X|\mathcal{G}]]$$

or in integral form $\int_A \mathbb{E}[X|\mathcal{G}](\omega)\, d\mathbb{P}(\omega) = \int_A X(\omega)\, d\mathbb{P}(\omega)$.