Proof. ($\Rightarrow$) This is a direct result of Lemma 2.2.1.

($\Leftarrow$) By tower property, it suffices to check that $M_n = \mathbb{E}(M_N|\mathcal{F}_n)$ for every $n \in \{0, 1, \ldots, N\}$. By assumption, we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_N)$ for any stopping time τ with values in $\{0, 1, \ldots, N\}$ (this is true since $\tau = N$ is also a stopping time). Let us fix t and consider an event $A \in \mathcal{F}_t$. Define τ_A as

$$\tau_A = \begin{cases} n, & \text{if } \omega \in A, \\ N, & \text{if } \omega \notin A, \end{cases}$$

then τ_A is an $\mathcal{F}$-stopping time with values in $\{0, 1, \ldots, N\}$ and so $\mathbb{E}(M_{\tau_A}) = \mathbb{E}(M_N)$. This yields $\mathbb{E}(\mathbf{1}_A M_n) = \mathbb{E}(\mathbf{1}_A M_N)$. Since this equality holds for any event $A \in \mathcal{F}_t$, by definition we have $M_n = \mathbb{E}(M_N|\mathcal{F}_n)$, which in turn implies that M is an $\mathcal{F}$-martingale. ∎