Proof. For (iii), we prove the case for an elementary process $\gamma \in \mathcal{R}$. Since the expected value of the cross terms are zero, i.e. $\mathbb{E}[(W_{t_{j+1}} - W_{t_j})(W_{t_{k+1}} - W_{t_k})] = 0$ for $j \neq k$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{j=0}^{m-1} \gamma_j (W_{t_{j+1}} - W_{t_j})\right)^2\right] &= \mathbb{E}\left[\sum_{j=0}^{m-1} \gamma_j^2 (W_{t_{j+1}} - W_{t_j})^2\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{m-1} \gamma_j^2 \mathbb{E}\left[(W_{t_{j+1}} - W_{t_j})^2 \mid \mathcal{F}_{t_j}\right]\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{m-1} \gamma_j^2 (t_{j+1} - t_j)\right] \\
&= \|\gamma\|_W^2
\end{aligned}$$

This means that $\hat{I}_T : (\mathcal{K}, \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is an isometry, i.e. a distance preserving transformation. This can be extended to the isometry $I_T : (L^2_{\mathbb{P}}(W), \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.[1]

For (iv), again we consider an elementary process γ and we can then do a similar proof to that of Theorem 3.3.2. We first compute the quadratic variation accumulated by the Ito integral on one of the subintervals $[t_j, t_{j+1}]$ on which $\gamma_u = \gamma_j$ is constant. For this, we choose partition points

$$t_j = s_0 < s_1 < \cdots < s_m = t_{j+1}$$

and consider

$$\sum_{i=0}^{m-1} (I_{s_{i+1}}(\gamma) - I_{s_i}(\gamma))^2 = \sum_{i=0}^{m-1} (\gamma_j (W_{s_{i+1}} - W_{s_i}))^2 = \gamma_j^2 \sum_{i=0}^{m-1} (W_{s_{i+1}} - W_{s_i})^2$$

As $m \to \infty$ and the step size $\max_{0 \leq i \leq m-1}(s_{i+1} - s_i)$ approaches zero, the term $\sum_{i=0}^{m-1}(W_{s_{i+1}} - W_{s_i})^2$ converges to the quadratic variation accumulated by Brownian motion between times t_j and t_{j+1}, which is $t_{j+1} - t_j$. Therefore, the limit of the above is

$$\gamma_j^2 (t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \gamma_u^2 \, du.$$

Adding up all these pieces for each of the subintervals $[t_j, t_{j+1}]$ completes the proof. ■

[1] The extension is done using the fact that the class of simple process $\mathcal{K}$ is a dense subset in the Banach space $L^2_{\mathbb{P}}(W)$ and the space $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is also complete (since it a Banach space or more specifically a Hilbert space).