

Complex Analysis

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calculus over $\mathbb{C}$ is not a cosmetic upgrade of **real analysis** — it is a different subject with better theorems. asking a function of a complex variable to be differentiable *once* forces it to be differentiable infinitely often, equal to its taylor series, and rigid enough that its values on a tiny arc determine it everywhere. the payoff for this rigidity: integrals that real methods cannot touch fall to a residue computation in three lines (Brown, James W. and Churchill, Ruel V., 2009).

1 the complex plane, quickly

$z = x + iy$ with $i^2 = -1$; conjugate $\bar{z} = x - iy$; modulus $|z|^2 = z\bar{z} = x^2 + y^2$; argument $\theta = \arg z$. the polar form runs through **euler’s formula**:

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta), \quad r = |z|, \quad (1)$$

which turns multiplication into “multiply moduli, add arguments” — complex multiplication *is* rotation-and-scaling of the plane. the n -th roots of unity $e^{2\pi i k/n}$, $k = 0, \dots, n-1$, sit at the vertices of a regular n -gon — the fact underneath the fft. note one loss: $\mathbb{C}$ has no useful order, so inequalities always live on moduli.

this is the same rotate-scale geometry as the svd page, compressed into a single number.

2 holomorphy and the cauchy–riemann equations

2.1 the definition

f is **holomorphic** at z_0 if

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (2)$$

exists — where h is *complex*, so the difference quotient must agree along every direction of approach. this is a far stronger demand than differentiability in $\mathbb{R}^2$, and it is where all the magic enters.

2.2 the derivation

write $f(x + iy) = u(x, y) + i v(x, y)$ and test two approach directions.

- **real approach** ($h = t \rightarrow 0$ along the real axis):

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \quad (3)$$

- **imaginary approach** ($h = it$, $t \rightarrow 0$):

$$f'(z_0) = \frac{u(x, y + t) + i v(x, y + t) - u(x, y) - i v(x, y)}{it} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \quad (4)$$

- **equate** real and imaginary parts:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad (5)$$

the **cauchy–riemann equations**: necessary for holomorphy, and together with continuity of the partials, sufficient. differentiating them once more gives $u_{xx} + u_{yy} = 0$ — both components of a holomorphic function are **harmonic**, which is why complex analysis moonlights as the theory of two-dimensional electrostatics and fluid flow.

3 contour integration

for a curve $\gamma : [a, b] \rightarrow \mathbb{C}$,

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt, \quad (6)$$

with the workhorse **ml-inequality**: $|\int_{\gamma} f dz| \leq ML$ where M bounds $|f|$ on γ and L is the arc length.

the one integral everyone must compute by hand — around the unit circle $\gamma(t) = e^{it}$:

$$\oint \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i, \quad \text{while} \quad \oint z^n dz = 0 \quad (n \neq -1). \quad (7)$$

every closed-contour integral in the subject is a bookkeeping of how many $\frac{1}{z}$ -type terms it encloses; the residue theorem below is exactly that bookkeeping, industrialised.

4 cauchy's theorem and integral formula

4.1 the theorem

cauchy's integral theorem. if f is holomorphic on and inside a simple closed contour γ , then

$$\oint_{\gamma} f(z) dz = 0. \quad (8)$$

proof sketch (with the extra assumption that f' is continuous): split into real contour integrals and apply **green's theorem**:

$$\begin{aligned} \oint_{\gamma} f dz &= \oint_{\gamma} (u dx - v dy) + i \oint_{\gamma} (v dx + u dy) \\ &= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dA = 0 + i0, \end{aligned}$$

both integrands vanishing by cauchy-riemann. ■ goursat later removed the continuity assumption entirely (triangle subdivision argument), which matters logically: it lets the theory bootstrap f' 's regularity from nothing.

4.2 the integral formula

for f holomorphic on and inside γ , and z_0 inside:

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz. \quad (9)$$

proof sketch: the integrand is holomorphic in the region between γ and a small circle C_{ε} around z_0 , so by cauchy's theorem the two integrals agree — deform γ down to C_{ε} . on C_{ε} , write $f(z) = f(z_0) + (f(z) - f(z_0))$: the first term contributes $f(z_0) \oint dz/(z - z_0) = 2\pi i f(z_0)$ by the unit-circle computation, and the second vanishes as $\varepsilon \rightarrow 0$ by the ml-inequality and continuity of f . ■

differentiating under the integral sign — legal, the kernel is smooth in z_0 — gives every derivative for free:

the theorem is really about topology: the integral over a contour only changes when the contour crosses a point where f fails to be holomorphic. contours slide freely through holomorphic territory.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad (10)$$

so a once-differentiable complex function is infinitely differentiable. no real-variable analogue of this exists.

5 consequences

- **cauchy estimates.** if $|f| \leq M$ on a circle of radius R about z_0 , the derivative formula with $|z - z_0| = R$ gives $|f^{(n)}(z_0)| \leq n! M/R^n$.
- **liouville's theorem, with proof.** an entire (holomorphic on all of $\mathbb{C}$) bounded function is constant: the $n = 1$ estimate gives $|f'(z_0)| \leq M/R$ for *every* R , so $f' \equiv 0$, so f is constant. ■
- **the fundamental theorem of algebra, free of charge.** let p be a nonconstant polynomial with no root. then $1/p$ is entire; and since $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, $1/p$ is bounded outside a large disc and (being continuous) on it. liouville forces $1/p$ constant — contradiction. every polynomial of degree $n \geq 1$ has a root, hence exactly n with multiplicity (Courant, Richard and Robbins, Herbert, 1941).
- **maximum modulus principle.** a nonconstant holomorphic function on a domain attains no interior maximum of $|f|$ — maxima live on the boundary. (via the mean value property: $f(z_0)$ is the average of f over any circle around z_0 , and an average cannot exceed its ingredients unless they are all equal.)
- **rouché's theorem** (the zero-counter): if $|g| < |f|$ on a closed contour, then f and $f + g$ have the same number of zeros inside. standard party trick: all five zeros of $z^5 + 3z + 1$ lie in $|z| < 2$, run the comparison with $f = z^5$ and $g = 3z + 1$ on $|z| = 2$: $|3z + 1| \leq 7 < 32 = |z^5|$, so $z^5 + 3z + 1$ has all five zeros inside.

an algebra theorem with no purely algebraic proof — every known proof smuggles in analysis or topology somewhere.

6 taylor and laurent series

- **taylor.** holomorphic on a disc $|z - z_0| < R$ means *equal* to the taylor series there:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad (11)$$

convergent on the largest disc avoiding singularities. (contrast real analysis, where e^{-1/x^2} is smooth yet equals no power series at 0.)

- **laurent.** holomorphic on an annulus $r < |z - z_0| < R$ means expandable with negative powers too:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (12)$$

the coefficient a_{-1} is the **residue** — the one term that survives closed-contour integration.

7 singularities, classified

an isolated singularity z_0 is judged by the principal (negative-power) part of the laurent expansion around it:

type	principal part	behaviour	example at 0
removable	none	f bounded near z_0 ; patchable	$\sin z/z$
pole, order m	finitely many terms	$ f \rightarrow \infty$; $(z - z_0)^m f$ removable	$1/z^2$
essential	infinitely many terms	wild: image dense in $\mathbb{C}$	$e^{1/z}$

riemann's removability theorem says bounded near z_0 already implies removable; casorati–weierstrass says an essential singularity's image is dense in every neighbourhood (picard sharpens: it hits every value but at most one, infinitely often).

$e^{1/z}$ near 0: along the positive real axis it explodes, along the negative it dies to 0, along the imaginary axis it spins on the unit circle. essential singularities have no manners.

8 the residue theorem

theorem. if f is holomorphic on and inside a positively-oriented simple closed contour γ except at finitely many interior points $z_1, \dots, z_k$, then

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f. \quad (13)$$

(shrink γ onto small circles around each singularity — cauchy's theorem kills everything else — and integrate each laurent series term by term; only a_{-1} survives.) computing residues:

- simple pole: $\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} (z - z_0)f(z)$; for $f = p/q$ with simple zero of q : $p(z_0)/q'(z_0)$.
- pole of order m : $\text{Res}_{z_0} f = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$.

8.1 a real integral, done honestly

claim. $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}.$

setup. $f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2}$: double poles at $\pm i$. integrate over the closed contour $\Gamma_R = \text{segment } [-R, R]$ plus the upper semicircle C_R (radius $R > 1$), which encloses only $z = i$.

residue at the double pole. with $m = 2$, $(z - i)^2 f(z) = (z + i)^{-2}$:

$$\text{Res}_{z=i} f = \lim_{z \rightarrow i} \frac{d}{dz} (z + i)^{-2} = \lim_{z \rightarrow i} \frac{-2}{(z + i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{-2}{-8i} = \frac{1}{4i} = -\frac{i}{4}. \quad (14)$$

apply the theorem.

$$\oint_{\Gamma_R} f(z) dz = 2\pi i \left(-\frac{i}{4} \right) = \frac{\pi}{2}. \quad (15)$$

kill the arc. on C_R , $|1 + z^2| \geq |z|^2 - 1 = R^2 - 1$, so $|f| \leq (R^2 - 1)^{-2}$, and by the ml-inequality

$$\left| \int_{C_R} f dz \right| \leq \frac{\pi R}{(R^2 - 1)^2} \xrightarrow{R \rightarrow \infty} 0. \quad (16)$$

conclude. the segment contribution converges to the real integral, the arc dies, and the contour total is pinned at $\frac{\pi}{2}$ for every $R > 1$:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}. \quad \blacksquare \quad (17)$$

the numeric check at the bottom of the page confirms both the integral and the residue. drilling this technique on a hundred variants is what (Schaum, 2009) is for.

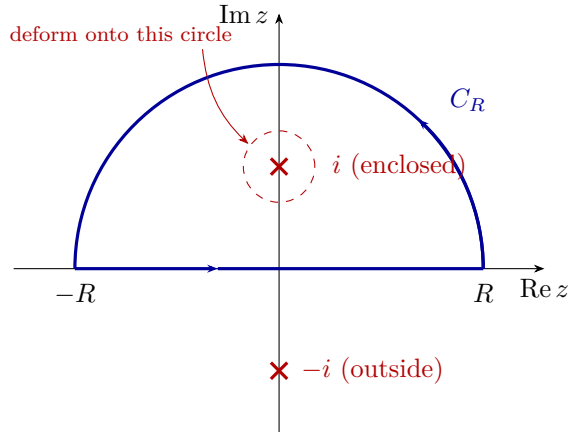


Figure 1: the semicircular contour Γ_R : the segment $[-R, R]$ picks up the real integral, the arc C_R dies like R^{-3} , and only the pole at i is enclosed. poles marked with crosses.

9 multivalued functions and branch cuts

$\log z = \ln|z| + i \arg z$ is the trouble: $\arg z$ is only defined up to $2\pi k$, so $\log$ (and with it $z^\alpha = e^{\alpha \log z}$, for non-integer α) is **multivalued**. the fix is surgical — choose a **branch cut**, a curve from the branch point 0 to ∞ that the function is forbidden to cross, and the remaining domain carries a single-valued holomorphic branch. the standard choice cuts along the negative real axis, keeping $\arg z \in (-\pi, \pi)$: approaching -1 from above gives $\log(-1) = i\pi$, from below $-i\pi$ — a jump of $2\pi i$ across the cut.

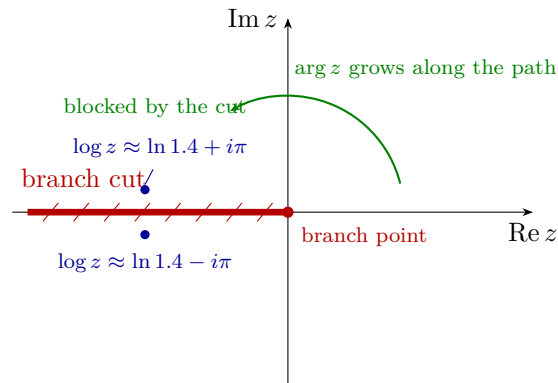


Figure 2: the principal branch of $\log z$: cut along the negative real axis, branch point at 0. the two test points straddling the cut disagree by $2\pi i$.

integrals with $\log$ or fractional powers in them use **keyhole contours** that hug both sides of the cut and collect the $2\pi i$ discrepancy as the answer — the cut stops being a nuisance and becomes the computational device.

10 conformal maps, a teaser

wherever $f'(z_0) \neq 0$, a holomorphic map is **conformal**: it preserves angles between intersecting curves (locally it multiplies by $f'(z_0)$ — a rotation-and-scale, which moves angles rigidly). the **riemann mapping theorem** is the astonishing global statement: every simply-connected proper subdomain of $\mathbb{C}$ — however jagged — is the image of the unit disc under some conformal bijection. the **mobius transformations** $z \mapsto \frac{az+b}{cz+d}$ ($ad - bc \neq 0$) are the conformal automorphisms of the riemann sphere, mapping circles-and-lines to circles-and-lines. applied mathematics uses all of this to teleport hard boundary-value problems onto discs, solve them there, and pull the answer back — see the harmonic-function remark under cauchy-riemann.

11 numerics: checking the residue computation

both claims of the worked example, verified: the real integral against $\pi/2$, and the residue at $z = i$ via a small numeric contour circle:

```
1 import numpy as np
2
3 # target: I = integral over R of dx / (1+x^2)^2 = pi/2 (via residue at z = i)
4 f = lambda x: 1.0 / (1.0 + x**2) ** 2
5
6 # (a) brute-force numeric integral on a huge interval
7 x = np.linspace(-2000, 2000, 8_000_001)
8 I_num = np.trapezoid(f(x), x)
9 print(f"numeric integral : {I_num:.10f}")
10 print(f"pi/2           : {np.pi/2:.10f}")
11
12 # (b) residue at the double pole z = i, computed numerically:
13 # res = (1/2 pi i) * contour integral of f around a small circle about i
14 t = np.linspace(0, 2 * np.pi, 400_001)
15 r = 0.05
16 z = 1j + r * np.exp(1j * t)           # circle around i
17 dz = 1j * r * np.exp(1j * t)
18 fz = 1.0 / (1.0 + z**2) ** 2
19 res = np.trapezoid(fz * dz, t) / (2j * np.pi)
20 print(f"numeric residue : {res:.10f}")
21 print(f"analytic residue : {-0.25j}")      # -i/4
22 print(f"2*pi*i*residue   : {(2j * np.pi * res).real:.10f} (should be pi/2)")
```

```
numeric integral : 1.5707963267
pi/2              : 1.5707963268
numeric residue   : 0.0000000000-0.2500000000j
analytic residue  : (-0-0.25j)
2*pi*i*residue    : 1.5707963268 (should be pi/2)
```

the numeric contour integral around a circle of radius 0.05 recovers the residue $-i/4$ to ten decimals — a direct demonstration that the residue is a property of the singularity alone, not of the contour used to probe it.

12 see also

- [real analysis](#) — the theory this page one-ups
- [single-variable calculus](#) — improper integrals, the honest way in
- [multivariable calculus](#) — green's theorem powers cauchy's

- **functional analysis** — operator theory keeps the resolvents complex
- **calculus** — the family home

references

Brown, James W. and Churchill, Ruel V. (2009). *Complex Variables and Applications*, McGraw-Hill Education.

Courant, Richard and Robbins, Herbert (1941). *What is Mathematics?*, Oxford University Press.

Schaum (2009). *Complex Variables with an Introduction to Conformal Mapping and Its Applications*, McGraw-Hill Education.