

Calculus

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calculus is the mathematics of change: it assigns exact meaning to two questions that stumped everyone from zeno to the seventeenth century — *how fast is a quantity changing right now?* and *how much of it has accumulated so far?* both questions are answered by the same primitive operation, the **limit**, applied in two different directions. everything on the pages below is a variation on that one move.

zeno's arrow: at every instant the arrow is motionless, so how does it move? calculus answers: velocity is a limit, not a snapshot.

1 the one idea: limits

every object in calculus is a limit in disguise. informally, $\lim_{x \rightarrow a} f(x) = L$ means: you can force $f(x)$ as close to L as you please by taking x close enough to a . the precise ε - δ formulation (and a worked proof) lives on the [single-variable page](#); the full logical foundations — why the real numbers support limits at all — belong to [real analysis](#).

from this primitive, two derived operations:

- the **derivative** is the limit of secant slopes:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}, \quad (1)$$

the instantaneous rate of change — geometrically, the slope of the tangent line that the secants converge to.

- the **integral** is the limit of rectangle sums as the mesh (the widest Δx_i) shrinks to zero:

$$\int_a^b f(x) dx = \lim_{\max_i \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i, \quad (2)$$

the accumulated total — geometrically, signed area under the curve.

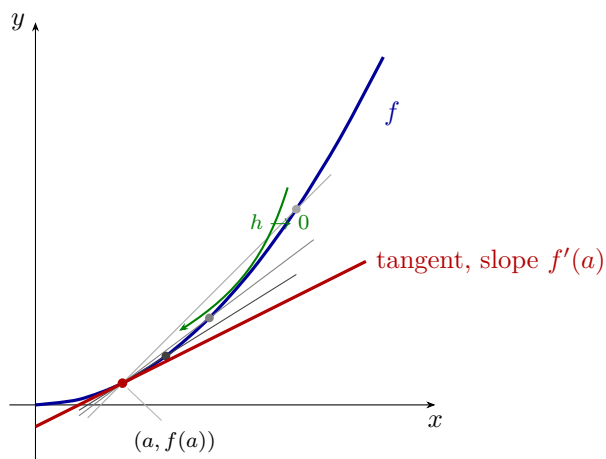


Figure 1: the derivative as a limit: secants through $(a, f(a))$ tilt down onto the tangent as the second point slides in.

2 two operations, one theorem

differentiation and integration look unrelated — one is local (a slope at a point), the other global (an area over an interval). the **fundamental theorem of calculus** says they are inverse operations, and it is the reason calculus is a calculus — a system of mechanical rules — rather than a bag of ad-hoc area tricks.

for f continuous on $[a, b]$:

- **part 1 (differentiation undoes accumulation).** the accumulation function $F(x) = \int_a^x f(t) dt$ is differentiable, and

$$F'(x) = f(x). \quad (3)$$

- **part 2 (antiderivatives compute integrals).** if Φ is *any* antiderivative of f (i.e. $\Phi' = f$), then

$$\int_a^b f(x) dx = \Phi(b) - \Phi(a). \quad (4)$$

the intuition for part 1 fits in one line: over a tiny window $[x, x+h]$ a continuous f is nearly constant, so the extra accumulated area is nearly a rectangle, $F(x+h) - F(x) \approx f(x)h$; divide by h and let $h \rightarrow 0$. part 2 is why the tables of derivatives you memorise pay double: read backwards, they are tables of integrals. proofs and the riemann-sum machinery behind the integral are developed on the [single-variable page](#).

before this theorem, every area was a bespoke exhaustion argument in the style of archimedes. after it, computing an area means guessing an antiderivative.

the theorem also scales up: in higher dimensions it reappears as the theorems of green, stokes and gauss — “the integral of a derivative over a region equals the data on the boundary” — the capstone of the [multivariable page](#).

3 the map

the child pages, in the order you should read them:

- [single-variable calculus](#) — limits made precise (ε - δ), continuity and its consequences (ivt, evt), differentiation rules with proofs, mean value theorem, taylor’s theorem with error control, l’hopital, riemann integration, both parts of the ftc, integration techniques, improper integrals.

- **multivariable calculus** — partial derivatives and the gradient, jacobians and hessians, taylor in several variables, unconstrained and constrained optimisation (lagrange multipliers), multiple integrals and change of variables, and vector calculus: div, curl, and the big three integral theorems.
- **differential equations** — where calculus earns its keep: equations whose unknowns are functions and whose vocabulary is derivatives. odes, solution methods, series solutions, and the frobenius machinery.

adjacent rooms in the wiki:

- **real analysis** is calculus with the safety off — the same theorems, but proved from the construction of the real numbers, plus the pathologies (nowhere-differentiable continuous functions) that show why the rigour is not optional.
- **complex analysis** is calculus over $\mathbb{C}$, where differentiability once means differentiability infinitely often and integrals compute themselves via residues.
- **optimisation** consumes the derivative wholesale: every gradient method is the linear approximation of this page applied iteratively.

4 a short history

newton had the essential ideas — his “method of fluxions” — by 1665–66, working privately during the plague years at woolsthorpe, but published almost nothing for decades. leibniz developed the subject independently around 1675 and published first (1684), contributing the notation that won: $\frac{dy}{dx}$ for the derivative and the elongated-s integral sign $\int$ for a sum of infinitesimal pieces. both men computed with “infinitesimals” — quantities smaller than any positive number yet not zero — which worked spectacularly and made no logical sense, as bishop berkeley gleefully pointed out in 1734, mocking them as “the ghosts of departed quantities”. the repair took a century and a half: cauchy (1821) reorganised the subject around limits, and weierstrass finally nailed the definition down in the ε – δ form used today. courant’s classic treatment (Courant, Richard and Robbins, Herbert, 1941) is still one of the best places to watch intuition and rigour being balanced honestly; his two-volume calculus (Courant, Richard, 1996) carries the same programme through several variables.

the priority dispute that followed was ugly enough that english mathematics, loyal to newton’s clumsier dot-notation, cut itself off from the continent for a century.

5 see also

- **single-variable calculus** — the core course
- **multivariable calculus** — the same ideas in $\mathbb{R}^n$
- **differential equations** — calculus applied to unknown functions
- **real analysis** — the rigorous foundations
- **functions** — the raw material

references

Courant, Richard (1996). *Differential and Integral Calculus*, Springer.

Courant, Richard and Robbins, Herbert (1941). *What is Mathematics?*, Oxford

University Press.