Solutions to Abbott's Understanding Analysis

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2026-09-21T09:00:00+10:00

Contents

Solutions to every exercise in Stephen Abbott's *Understanding Analysis* (2nd edition, Springer UTM, 2015) — 467 exercises across chapters 1–8, from the construction of the real numbers to Fourier series. The book is filed at Understanding Analysis.

1 The Real Numbers

Contents

1.1 Exercises 1.2.1–1.2.7

Problem 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution. (a) Both are irrational, and in each case the proof of Theorem 1.1.1 runs unchanged except for the divisibility lemma that drives it.
For $\sqrt{3}$ the lemma needed is that $3 \mid p^2$ implies $3 \mid p$: writing $p = 3k + s$ with $s \in \{0, 1, 2\}$,

$$p^2 = 3(3k^2 + 2ks) + s^2, \qquad s^2 \in \{0, 1, 4\},$$

so p^2 leaves remainder 0 on division by 3 only when $s = 0$.
Now assume for contradiction that $\sqrt{3} = p/q$ with $p, q \in \mathbf{Z}$ having no common factor. Then

$$\begin{aligned} p^2 &= 3q^2 &&\Rightarrow\ 3 \mid p, \text{ say } p = 3r, \\ 9r^2 &= 3q^2 &&\Rightarrow\ q^2 = 3r^2 \ \Rightarrow\ 3 \mid q, \end{aligned}$$

so 3 divides both p and q, contradicting that the fraction is in lowest terms.
For $\sqrt{6}$ the same strategy works, now run on the prime 2. Assume $\sqrt{6} = p/q$ in lowest terms, so $p^2 = 6q^2$. Then p^2 is even, hence p is even (the square of an odd number is odd); writing $p = 2r$ gives $4r^2 = 6q^2$, that is,

$$2r^2 = 3q^2.$$

The left side is even, and 3 is odd, so q^2 is even and therefore q is even. Again p and q share the factor 2, a contradiction.
(b) It breaks at the final substitution, where the contradiction is supposed to appear. Assume $\sqrt{4} = p/q$ in lowest terms, so $p^2 = 4q^2$. As before p^2 is even, so $p = 2r$ for some integer r, and substituting yields

$$4r^2 = 4q^2, \qquad \text{i.e.} \qquad r^2 = q^2.$$

In Theorem 1.1.1 the corresponding substitution left $2r^2 = q^2$, forcing q^2, and hence q, to be even. Here the 4 on the right cancels the 4 produced by $p = 2r$ exactly, and $r^2 = q^2$ says nothing about the parity of q, so no contradiction is reachable.

Problem 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Clearing the denominator of r turns $2^r = 3$ into an even integer equal to an odd one.
Assume for contradiction that $r = p/q$ with $p, q \in \mathbf{Z}$ and $q \geq 1$. Raising $2^{p/q} = 3$ to the q-th power gives

$$2^p = \left(2^{p/q}\right)^q = 3^q.$$

If $p \leq 0$ then $2^p \leq 1 < 3 \leq 3^q$, so in fact $p \geq 1$ and both sides are positive integers. The left side is then even; the right side is a product of $q \geq 1$ odd numbers, hence odd. An integer cannot be both, so no such r exists.

Problem 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.
(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution. (a) False, (b) true, (c) false, (d) true, (e) true.
(a) Take the tails of $\mathbf{N}$:

$$A_n = \{n, n+1, n+2, \ldots\}.$$

Each A_n is infinite and $A_1 \supseteq A_2 \supseteq \cdots$, but $\bigcap_{n=1}^{\infty} A_n = \emptyset$, since any $m \in \mathbf{N}$ fails to lie in A_{m+1}.
(b) True. Because $A_n \subseteq A_1$ and A_1 is finite, the cardinalities

$$|A_1| \;\geq\; |A_2| \;\geq\; |A_3| \;\geq\; \cdots \;\geq\; 1$$

form a nonincreasing sequence of positive integers, which must therefore be eventually constant: there is an N with $|A_n| = |A_N|$ for all $n \geq N$. For such n the inclusion $A_n \subseteq A_N$ between finite sets of equal cardinality forces $A_n = A_N$. Hence

$$\bigcap_{n=1}^{\infty} A_n \;=\; \bigcap_{n=1}^{N} A_n \;=\; A_N,$$

the last equality by the nesting. So the intersection equals A_N, which is finite and nonempty.
(c) False. Take $A = B = \{1\}$ and $C = \{2\}$:

$$\begin{aligned} A \cap (B \cup C) &= \{1\} \cap \{1,2\} = \{1\}, \\ (A \cap B) \cup C &= \{1\} \cup \{2\} = \{1,2\}. \end{aligned}$$

(d) True, and it is just associativity of $\cap$: each side consists precisely of those x belonging to all three of A, B, C, since

$$\begin{aligned} x \in A \cap (B \cap C) &\iff x \in A \;\wedge\; (x \in B \;\wedge\; x \in C) \\ &\iff (x \in A \;\wedge\; x \in B) \;\wedge\; x \in C \\ &\iff x \in (A \cap B) \cap C. \end{aligned}$$

(e) True, and it is the distributive law for $\wedge$ over $\vee$:

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \;\wedge\; (x \in B \;\vee\; x \in C) \\ &\iff (x \in A \wedge x \in B) \;\vee\; (x \in A \wedge x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Problem 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Sort each natural number by the power of 2 it contains: set

$$A_i \;=\; \left\{\, 2^{\,i-1}(2k-1) \;:\; k \in \mathbf{N} \,\right\},$$

so that $A_1 = \{1, 3, 5, 7, \ldots\}$, $A_2 = \{2, 6, 10, 14, \ldots\}$, $A_3 = \{4, 12, 20, 28, \ldots\}$, and in general A_i consists of the natural numbers whose largest power-of-two divisor is exactly 2^{i-1}.
Each A_i is infinite because $k \mapsto 2^{\,i-1}(2k-1)$ is injective on $\mathbf{N}$. The other two properties are both the statement that every $n \in \mathbf{N}$ can be written as

$$n = 2^a m, \qquad a \geq 0,\ m \text{ odd},$$

in exactly one way. Existence: divide n by 2 repeatedly until an odd quotient m appears, which happens after finitely many steps since the quotients strictly decrease. Uniqueness: if $2^a m = 2^b m_1$ with $a < b$ and m, m_1 odd, then $m = 2^{\,b-a} m_1$ is even, a contradiction; so $a = b$ and then $m = m_1$.
Consequently n lies in A_{a+1} and in no other A_i, which gives simultaneously $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Problem 1.2.5. (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.
(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution. Because failing a conjunction is failing one of its two clauses, and failing a disjunction is failing both.
(a) Let $x \in (A \cap B)^c$, and suppose x lies in neither A^c nor B^c. Then $x \in A$ and $x \in B$, so

$$x \in A \cap B,$$

contradicting $x \notin A \cap B$. Hence $x \in A^c$ or $x \in B^c$, that is, $x \in A^c \cup B^c$; as x was arbitrary, $(A \cap B)^c \subseteq A^c \cup B^c$.
(b) Let $x \in A^c \cup B^c$, so $x \notin A$ or $x \notin B$. Membership in $A \cap B$ demands both, so in either case $x \notin A \cap B$, that is, $x \in (A \cap B)^c$. Hence $A^c \cup B^c \subseteq (A \cap B)^c$, and combining with (a),

$$(A \cap B)^c = A^c \cup B^c.$$

(c) Both inclusions.
($\subseteq$) If $x \in (A \cup B)^c$ then $x \notin A \cup B$; since $x \in A$ alone would already put x in $A \cup B$, and likewise for B, we get $x \notin A$ and $x \notin B$, i.e. $x \in A^c \cap B^c$.
($\supseteq$) If $x \in A^c \cap B^c$ then $x \notin A$ and $x \notin B$. Membership in $A \cup B$ would require x to lie in at least one of them, so $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$.
Therefore $(A \cup B)^c = A^c \cap B^c$.

Problem 1.2.6. (a) Verify the triangle inequality in the special case where a and b have the same sign.
(b) Find an efficient proof for all the cases at once by first demonstrating $(a + b)^2 \leq (|a| + |b|)^2$.
(c) Prove $|a - b| \leq |a - c| + |c - d| + |d - b|$ for all a, b, c, and d.
(d) Prove $\big||a| - |b|\big| \leq |a - b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution. (a) When a and b share a sign the triangle inequality holds with equality.
(i) $a \geq 0$ and $b \geq 0$. Then $a + b \geq 0$, so

$$|a + b| = a + b = |a| + |b|.$$

(ii) $a \leq 0$ and $b \leq 0$. Then $a + b \leq 0$, so

$$|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|.$$

(b) Every real x satisfies $x \leq |x|$ and $x^2 = |x|^2$, and $|ab| = |a||b|$ by property (i) of Example 1.2.5, whose sign-by-sign verification is independent of the property (ii) being proved here. Hence $ab \leq |ab| = |a||b|$, and

$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= \big(|a| + |b|\big)^2.\end{aligned}$$

Now put $s = |a + b| \geq 0$ and $t = |a| + |b| \geq 0$. The display says $s^2 = (a + b)^2 \leq t^2$, and for nonnegative numbers $s > t$ would force $s^2 > t^2$; therefore $s \leq t$, which is

$$|a + b| \leq |a| + |b|.$$

(c) Insert c and d and apply (b) twice:

$$\begin{aligned}|a - b| &= \big|(a - c) + (c - d) + (d - b)\big| \\ &\leq |a - c| + \big|(c - d) + (d - b)\big| \\ &\leq |a - c| + |c - d| + |d - b|.\end{aligned}$$

(d) Apply (b) to the identity $a = (a - b) + b$, and then to $b = (b - a) + a$:

$$|a| \le |a-b| + |b| \;\Rightarrow\; |a| - |b| \le |a-b|,$$
$$|b| \le |b-a| + |a| \;\Rightarrow\; |b| - |a| \le |b-a| = |a-b|.$$

The number $\big||a| - |b|\big|$ equals $|a| - |b|$ or $|b| - |a|$ according to the sign of $|a| - |b|$, and both are bounded by $|a - b|$; hence $\big||a| - |b|\big| \le |a - b|$.

Problem 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.
(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \le x \le 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
(b) Find two sets A and B for which $f(A \cap B) \ne f(A) \cap f(B)$.
(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.
(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution. (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$; both equalities hold in this case.
Since $x \mapsto x^2$ is increasing on $[0, \infty)$ and both intervals lie there, it carries each interval onto the interval between the squares of its endpoints. With $A \cap B = [1, 2]$ and $A \cup B = [0, 4]$,

$$f(A \cap B) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B),$$
$$f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B).$$

(b) Take $A = \{1\}$ and $B = \{-1\}$, still with $f(x) = x^2$. Then $A \cap B = \emptyset$, so

$$f(A \cap B) = \emptyset \ne \{1\} = \{1\} \cap \{1\} = f(A) \cap f(B).$$

(c) Let $y \in g(A \cap B)$, say $y = g(x)$ with $x \in A \cap B$. Then $x \in A$ gives $y \in g(A)$, and $x \in B$ gives $y \in g(B)$; hence $y \in g(A) \cap g(B)$.
(d) Conjecture: $g(A \cup B) = g(A) \cup g(B)$ for every g and all $A, B \subseteq \mathbf{R}$. Unlike in (c), a single point of $A \cup B$ already lies in one of the two sets, so every step reverses:

$$\begin{aligned} y \in g(A \cup B) &\iff y = g(x) \text{ for some } x \in A \cup B \\ &\iff y = g(x) \text{ for some } x \in A, \\ &\qquad \text{or } y = g(x) \text{ for some } x \in B \\ &\iff y \in g(A) \;\vee\; y \in g(B) \\ &\iff y \in g(A) \cup g(B). \end{aligned}$$

1.2 Exercises 1.2.8–1.3.1

Problem 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1–1) if $a_1 \ne a_2$ in A implies that $f(a_1) \ne f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.
Give an example of each or state that the request is impossible:
(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1–1 but not onto.
(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1–1.
(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1–1 and onto.

Solution. All three are possible. (Throughout, $\mathbf{N} = \{1, 2, 3, \ldots\}$.)
(a) $f(n) = n + 1$. It is 1–1 since $f(a_1) = f(a_2)$ forces $a_1 = a_2$, and it is not onto because $f(n) \ge 2$ for every n, so 1 has no preimage.

(b)

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ n - 1 & \text{if } n \geq 2. \end{cases}$$

Onto: given $m \in \mathbf{N}$, $f(m+1) = m$. Not 1–1: $f(1) = f(2) = 1$.
(c) Interleave the nonnegative and negative integers:

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd}, \\ -n/2 & \text{if } n \text{ is even}, \end{cases}$$

so that $(f(1), f(2), f(3), f(4), \ldots) = (0, -1, 1, -2, \ldots)$. The map

$$g(m) = \begin{cases} 2m + 1 & \text{if } m \geq 0, \\ -2m & \text{if } m < 0 \end{cases}$$

sends $\mathbf{Z}$ into $\mathbf{N}$ and satisfies $f(g(m)) = m$ and $g(f(n)) = n$ (Check!), the branches never crossing because odd n give $f(n) \geq 0$ and even n give $f(n) < 0$. A map with a two-sided inverse is 1–1 and onto.

Problem 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.
(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution. (a) $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$: since $x^2 \geq 0$ always, $x^2 \in [0, 4]$ says exactly $|x| \leq 2$, and $x^2 \in [-1, 1]$ says exactly $|x| \leq 1$. Both identities hold here, with $A \cap B = [0, 1]$ and $A \cup B = [-1, 4]$:

$$\begin{aligned} f^{-1}(A \cap B) &= \{x : 0 \leq x^2 \leq 1\} = [-1, 1] = f^{-1}(A) \cap f^{-1}(B), \\ f^{-1}(A \cup B) &= \{x : -1 \leq x^2 \leq 4\} = [-2, 2] = f^{-1}(A) \cup f^{-1}(B). \end{aligned}$$

(b) Each identity is one chain of equivalences, since $x \in g^{-1}(S)$ just relocates the membership test to $g(x) \in S$. For every $x \in \mathbf{R}$,

$$\begin{aligned} x \in g^{-1}(A \cap B) &\iff g(x) \in A \cap B \\ &\iff g(x) \in A \ \wedge \ g(x) \in B \\ &\iff x \in g^{-1}(A) \ \wedge \ x \in g^{-1}(B) \\ &\iff x \in g^{-1}(A) \cap g^{-1}(B), \end{aligned}$$

and verbatim with $\cap$ replaced by $\cup$ and $\wedge$ by $\vee$:

$$\begin{aligned} x \in g^{-1}(A \cup B) &\iff g(x) \in A \ \vee \ g(x) \in B \\ &\iff x \in g^{-1}(A) \ \vee \ x \in g^{-1}(B) \\ &\iff x \in g^{-1}(A) \cup g^{-1}(B). \end{aligned}$$

Problem 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
(c) Two real numbers satisfy $a \le b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution. (a) False, (b) False, (c) True.
(a) The forward implication is fine ($a < b \le b + \epsilon$ for $\epsilon > 0$), but the converse fails: take $a = b = 0$. Then $a < b + \epsilon$ holds for every $\epsilon > 0$, while $a < b$ does not.
(b) This is precisely the failing converse just exhibited: $a = b = 0$ satisfies the hypothesis and violates the conclusion.
(c) ($\Rightarrow$) If $a \le b$ then $a \le b < b + \epsilon$ for every $\epsilon > 0$.
($\Leftarrow$) Contrapositively, suppose $a > b$ and put $\epsilon = a - b > 0$. Then

$$b + \epsilon = b + (a - b) = a,$$

so $a < b + \epsilon$ fails for that one ϵ. Hence if the inequality holds for *every* $\epsilon > 0$, we must have $a \le b$.

Problem 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.
(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.
(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.
(c) Between every two distinct real numbers there is a rational number.

Solution. (a) Negation: *There exist real numbers $a < b$ such that $a + 1/n \ge b$ for every $n \in \mathbf{N}$.* Symbolically the claim reads

$$(\forall a < b)(\exists n \in \mathbf{N})\ a + 1/n < b,$$

so its negation is $(\exists a < b)(\forall n \in \mathbf{N})\ a + 1/n \ge b$. The claim is the true one: it says $1/n$ can be made smaller than the gap $b - a > 0$, which is the Archimedean Property (Theorem 1.4.2 (ii)).
(b) Negation: *For every real number $x > 0$ there exists an $n \in \mathbf{N}$ satisfying $1/n \le x$.* The negation is the true one: it is Theorem 1.4.2 (ii) again, applied with $y = x$.
(c) Negation: *There exist two distinct real numbers $a < b$ such that every rational number r satisfies $r \le a$ or $r \ge b$.* The claim is the true one: it is the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).

Problem 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.
(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.
(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution. (a) Base case: $y_1 = 6 > -6$. Assuming $y_n > -6$,

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = \frac{-18}{3} = -6,$$

the strict inequality surviving because $t \mapsto (2t - 6)/3$ is increasing. By induction $y_n > -6$ for all $n \in \mathbf{N}$.
(b) The recursion is affine, so consecutive differences contract by the factor $2/3$:

$$y_{n+2} - y_{n+1} = \frac{2y_{n+1} - 6}{3} - \frac{2y_n - 6}{3} = \frac{2}{3}\,(y_{n+1} - y_n).$$

Base case: $y_2 = (2 \cdot 6 - 6)/3 = 2 < 6 = y_1$. Assuming $y_{n+1} < y_n$, the display gives $y_{n+2} - y_{n+1} = \frac{2}{3}(y_{n+1} - y_n) < 0$, i.e. $y_{n+2} < y_{n+1}$. By induction $y_{n+1} < y_n$ for every $n \in \mathbf{N}$, so (y_n) is decreasing.

Problem 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.
(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.
(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.
(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution. (a) Induct on n, the two-set law $(A \cup B)^c = A^c \cap B^c$ of Exercise 1.2.5 (c) supplying both the base case ($n = 2$; the case $n = 1$ is the identity $A_1^c = A_1^c$) and the inductive step. Assume the statement for some $n \geq 2$. Grouping the first n sets as a single set and applying Exercise 1.2.5 (c) to that set and A_{n+1},

$$\begin{aligned}(A_1 \cup \cdots \cup A_n \cup A_{n+1})^c &= \big((A_1 \cup \cdots \cup A_n) \cup A_{n+1}\big)^c \\ &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c,\end{aligned}$$

the last step by the induction hypothesis. This is the statement for $n+1$, so it holds for all $n \in \mathbf{N}$.
(b) Take the tails of $\mathbf{N}$: $B_i = \{i, i+1, i+2, \ldots\}$. These are nested ($B_1 \supseteq B_2 \supseteq \cdots$), so for each n

$$\bigcap_{i=1}^{n} B_i = B_n \ni n, \qquad \text{hence } \bigcap_{i=1}^{n} B_i \neq \emptyset.$$

But $\bigcap_{i=1}^{\infty} B_i = \emptyset$: any candidate $m \in \mathbf{N}$ satisfies $m < m+1$, so $m \notin B_{m+1}$.
(c) One chain of equivalences, driven by the quantifier duality $\neg\exists = \forall\neg$, which is indifferent to the size of the index set. For any x,

$$\begin{aligned} x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c &\iff \neg\big(\exists\, i \in \mathbf{N} : x \in A_i\big) \\ &\iff \forall\, i \in \mathbf{N},\ x \notin A_i \\ &\iff \forall\, i \in \mathbf{N},\ x \in A_i^c \\ &\iff x \in \bigcap_{i=1}^{\infty} A_i^c. \end{aligned}$$

Problem 1.3.1. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.
(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution. (a) A real number i is the **greatest lower bound** (or **infimum**) for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:
(i) i is a lower bound for A;
(ii) if l is any lower bound for A, then $l \leq i$.
We write $i = \inf A$.

(b) **Lemma.** Assume $i \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $i = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < i + \epsilon$.
($\Rightarrow$) Assume $i = \inf A$ and let $\epsilon > 0$. Since $i < i + \epsilon$, criterion (ii) shows $i + \epsilon$ is not a lower bound for A; that is, some $a \in A$ satisfies $a < i + \epsilon$.
($\Leftarrow$) Let l be any lower bound for A and suppose, for contradiction, that $i < l$. Taking $\epsilon = l - i > 0$ produces $a \in A$ with

$$a < i + \epsilon = l,$$

contradicting the fact that l is a lower bound. Hence $l \leq i$, which is criterion (ii), and $i = \inf A$.

1.3 Exercises 1.3.2–1.3.8

Problem 1.3.2. Give an example of each of the following, or state that the request is impossible.
(a) A set B with $\inf B \geq \sup B$.
(b) A finite set that contains its infimum but not its supremum.
(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution. (a) $B = \{3\}$, for which $\inf B = \sup B = 3$. Equality is the only way this can happen: if B is nonempty, picking any $b \in B$ gives

$$\inf B \leq b \leq \sup B,$$

so the strict inequality $\inf B > \sup B$ is impossible.
(b) Impossible. A finite nonempty set $F = \{x_1, \ldots, x_n\}$ contains a largest element x_M (induction on n: for $n = 1$ take x_1, and otherwise take the larger of x_n and the maximum of $\{x_1, \ldots, x_{n-1}\}$). That x_M is an upper bound lying in F, so $x_M = \sup F$ by Exercise 1.3.7, and F contains its supremum. (The empty set has no infimum to contain either.)
(c) $B = \{1/n : n \in \mathbf{N}\} \subseteq \mathbf{Q}$, the set of Example 1.3.3. It is bounded, $1 \in B$ is an upper bound so $\sup B = 1 \in B$ by Exercise 1.3.7, and $\inf B = 0 \notin B$: zero is a lower bound, and no $\epsilon > 0$ is, because the Archimedean Property (Theorem 1.4.2) supplies $n \in \mathbf{N}$ with $1/n < \epsilon$.

Problem 1.3.3. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.
(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. (a) The Axiom of Completeness applies to B: it is nonempty because A is bounded below, and it is bounded above because every $a \in A$ is an upper bound for B (each $b \in B$ satisfies $b \leq a$ by definition of lower bound). So $s = \sup B$ exists; we check the two criteria of Exercise 1.3.1(a) for $s = \inf A$.
(i) s is a lower bound for A. Fix $a \in A$. As just noted a is an upper bound for B, so Definition 1.3.2(ii) gives

$$s \leq a \qquad \text{for every } a \in A.$$

(ii) If l is any lower bound for A, then $l \in B$ by the definition of B, and s is an upper bound for B, so $l \leq s$.
Hence $\sup B = s = \inf A$.
(b) Part (a) **derives** the existence of greatest lower bounds from the Axiom of Completeness: given any nonempty A bounded below, the set B of its lower bounds is nonempty and bounded above, the Axiom produces $\sup B$, and (a) identifies that number as $\inf A$. An assertion provable from the axiom already stated is a theorem, not an axiom, so adding it would be redundant.

Problem 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.
(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution. (a) $\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\}$. Write $s_k = \sup A_k$ and $s = \max\{s_1, s_2\}$ (a maximum of two reals always exists), and verify Definition 1.3.2.
(i) Any $x \in A_1 \cup A_2$ lies in some A_k, whence $x \leq s_k \leq s$.
(ii) If b is an upper bound for $A_1 \cup A_2$, it is in particular an upper bound for each A_k, so $s_k \leq b$ for $k = 1, 2$ and therefore $s = \max\{s_1, s_2\} \leq b$.
Induction extends this to any finite union: assuming the formula for $n-1$ sets and applying the two-set case to $\left(\bigcup_{k=1}^{n-1} A_k\right) \cup A_n$,

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max\{\sup A_1, \dots, \sup A_n\}.$$

(b) No. The union of infinitely many bounded sets need not be bounded above, and even when it is, the set of suprema need not have a maximum. Both failures are visible at once:
(i) $A_k = \{k\}$ gives $\bigcup_{k=1}^{\infty} A_k = \mathbf{N}$, which has no supremum at all (Theorem 1.4.2).
(ii) $A_k = \{1 - 1/k\}$ gives a bounded union whose suprema $1 - 1/k$ have no largest element, so $\max_k \sup A_k$ does not exist; nevertheless $\sup\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$.
The correct extension replaces the maximum by a supremum: if $\bigcup_{k=1}^{\infty} A_k$ is bounded above, then so is $S = \{\sup A_k : k \in \mathbf{N}\}$ (any upper bound b for the union bounds each A_k, hence each $\sup A_k$), and

$$\sup\left(\bigcup_{k=1}^{\infty} A_k\right) = \sup S.$$

Indeed $\sup S$ is an upper bound for the union, since $x \in A_k$ forces $x \leq \sup A_k \leq \sup S$; and any upper bound b for the union satisfies $\sup A_k \leq b$ for every k, so $\sup S \leq b$.

Problem 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.
(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.
(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution. (a) Set $s = \sup A$. If $c = 0$ then $cA = \{0\}$ and both sides equal 0, so assume $c > 0$ and check the two parts of Definition 1.3.2.
(i) For every $a \in A$ we have $a \leq s$, and multiplying by $c > 0$ preserves the inequality:

$$ca \leq cs \qquad \text{for all } a \in A,$$

so cs is an upper bound for cA.
(ii) Let b be an arbitrary upper bound for cA, so $ca \leq b$ for all $a \in A$. Dividing by $c > 0$ gives $a \leq b/c$ for all $a \in A$, i.e. b/c is an upper bound for A. Because s is the **least** upper bound, $s \leq b/c$, and multiplying by $c > 0$ yields $cs \leq b$.
Hence $\sup(cA) = cs = c \sup A$.
(b) For $c < 0$, $\sup(cA) = c \inf A$ whenever A is also bounded below. Multiplication by c reverses inequalities, so it trades suprema for infima, and the identity available from the standing hypothesis (bounded above only) is the reflected one:

$$\inf(cA) = c \sup A.$$

That one holds by the criteria of Exercise 1.3.1(a) with $s = \sup A$. Every $a \in A$ satisfies $a \leq s$, hence $ca \geq cs$, so cs is a lower bound for cA. If l is any lower bound for cA, then $ca \geq l$ for all $a \in A$ gives $a \leq l/c$, so l/c is an upper bound for A, so $s \leq l/c$ and therefore $cs \geq l$. And $\sup(cA) = c \inf A$ follows by applying this identity to the set cA with the scalar $1/c < 0$, legitimate because A bounded below makes cA bounded above.

Problem 1.3.6. Given sets A and B, define $A+B = \{a+b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A+B) = \sup A + \sup B$.
(a) Let $s = \sup A$ and $t = \sup B$. Show $s+t$ is an upper bound for $A+B$.
(b) Now let u be an arbitrary upper bound for $A+B$, and temporarily fix $a \in A$. Show $t \leq u-a$.
(c) Finally, show $\sup(A+B) = s+t$.
(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution. (a) A typical element of $A+B$ is $a+b$ with $a \in A$ and $b \in B$, and adding the inequalities $a \leq s$ and $b \leq t$ gives

$$a+b \leq s+t.$$

(b) With $a \in A$ fixed, $a+b \in A+B$ for every $b \in B$, so $a+b \leq u$, i.e.

$$b \leq u-a \qquad \text{for all } b \in B.$$

Thus $u-a$ is an upper bound for B, and since t is the least upper bound, $t \leq u-a$.
(c) The inequality from (b) holds for **every** $a \in A$ (the choice of a was arbitrary), and it rearranges to $a \leq u-t$. So $u-t$ is an upper bound for A, whence $s \leq u-t$, i.e.

$$s+t \leq u.$$

By (a), $s+t$ is an upper bound for $A+B$; by the display, it is no larger than any other upper bound u. Both criteria of Definition 1.3.2 hold, so $\sup(A+B) = s+t$.
(d) **Method (2):** $s+t$ is an upper bound for $A+B$ by (a), so Lemma 1.3.8 reduces the claim to producing, for each $\epsilon > 0$, an element of $A+B$ exceeding $(s+t)-\epsilon$. Applying the lemma to A and to B with $\epsilon/2$ gives $a \in A$ and $b \in B$ with $s - \epsilon/2 < a$ and $t - \epsilon/2 < b$; adding,

$$(s+t) - \epsilon < a+b \in A+B.$$

Lemma 1.3.8 now yields $\sup(A+B) = s+t$.

Problem 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. Criterion (i) of Definition 1.3.2 is the hypothesis that a is an upper bound for A, so only (ii) needs checking. Let b be any upper bound for A. Since $a \in A$, the defining property of an upper bound applied to the element a gives

$$a \leq b.$$

Thus a is an upper bound no larger than any other, and $a = \sup A$.

Problem 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:
(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.
(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.
(c) $\{n/(3n+1) : n \in \mathbf{N}\}$.
(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$.

Solution. (a) $\sup = 1$ and $\inf = 0$, neither attained: the set is exactly $\mathbf{Q} \cap (0,1)$.
(b) $\sup = 1$ and $\inf = -1$, both attained at $n = 1$ (m even, resp. m odd), hence a maximum and a minimum.
(c) $\sup = 1/3$, not attained; $\inf = 1/4$, attained at $n = 1$, since $n/(3n+1) = 1/(3+1/n)$ increases in n.
(d) $\sup = 1$ and $\inf = 0$, neither attained, since $m/(m+n) = 1/(1+n/m)$ with n/m ranging over all

positive rationals.

1.4 Exercises 1.3.9–1.4.4

Problem 1.3.9. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not always the case if we only assume $\sup A \le \sup B$.

Solution. (a) Apply Lemma 1.3.8 to B with $\epsilon = \sup B - \sup A > 0$ (legitimate precisely because the inequality is strict): there is $b \in B$ with

$$\sup A = \sup B - \epsilon < b.$$

Then every $a \in A$ satisfies $a \le \sup A < b$, so b is an upper bound for A.
(b) Take $A = \{0\}$ and $B = \{-1/n : n \in \mathbf{N}\}$. Here $\sup A = 0 = \sup B$, yet every $b \in B$ is negative and hence fails to bound $0 \in A$.

Problem 1.3.10. (Cut Property). The Cut Property of the real numbers is the following:
If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \le c$ whenever $x \in A$ and $x \ge c$ whenever $x \in B$.
(a) Use the Axiom of Completeness to prove the Cut Property.
(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.
(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution. (a) Take $c = \sup A$. It exists by the Axiom of Completeness: A is nonempty, and each of the (at least one) elements $b \in B$ is an upper bound for A because $a < b$ for every $a \in A$. The two required properties are then exactly the two halves of Definition 1.3.2:
(i) $x \le c$ for all $x \in A$, since c is an upper bound for A;
(ii) $c \le x$ for all $x \in B$, since each such x is an upper bound for A and c is the least one.
(b) Split $\mathbf{R}$ at E: let

$$\begin{aligned} B &= \{b \in \mathbf{R} : b \text{ is an upper bound for } E\}, \\ A &= \mathbf{R} \setminus B. \end{aligned}$$

These are disjoint with union $\mathbf{R}$ by construction; $B \ne \emptyset$ because E is bounded above, and $A \ne \emptyset$ because picking $x_0 \in E$ (possible as $E \ne \emptyset$) makes $x_0 - 1$ fail to be an upper bound. If $a \in A$ and $b \in B$, then a is not an upper bound, so some $x \in E$ has $x > a$, whence $b \ge x > a$. The Cut Property supplies c with $x \le c$ on A and $c \le x$ on B.
Then $c = \sup E$. For (i), if c were not an upper bound for E there would be $x \in E$ with $x > c$; the midpoint $t = (c + x)/2$ satisfies $t < x$, so t is not an upper bound, i.e. $t \in A$, forcing $t \le c$ and contradicting $t > c$. For (ii), any upper bound b for E lies in B, so $c \le b$.
(c) Cut $\mathbf{Q}$ at $\sqrt{2}$:

$$\begin{aligned} A &= \{x \in \mathbf{Q} : x \le 0\} \cup \{x \in \mathbf{Q} : x > 0,\ x^2 < 2\}, \\ B &= \{x \in \mathbf{Q} : x > 0,\ x^2 > 2\}. \end{aligned}$$

They are nonempty ($1 \in A$, $2 \in B$), disjoint, and $A \cup B = \mathbf{Q}$ because no rational satisfies $x^2 = 2$ (Theorem 1.1.1); and $a < b$ for $a \in A$, $b \in B$, since $a \le 0 < b$ in the first case and $a^2 < 2 < b^2$ with both positive in the second. A cut point c would lie in A or in B, and would be a maximum of A or a minimum of B respectively. Neither exists: for $c \in A$ with $c > 0$, choose $n \in \mathbf{N}$ with $n > (2c + 1)/(2 - c^2)$ (a rational bound, so such an n is found by inspection of its numerator, with no appeal to completeness), so that

$$\left(c+\tfrac{1}{n}\right)^2 \le c^2 + \frac{2c+1}{n} < 2$$

and $c+1/n \in A$ exceeds c (and if $c \le 0$, already $1 \in A$ exceeds c); for $c \in B$, choose $n > \max\{2c/(c^2 - 2),\ 1/c\}$, so that $c - 1/n > 0$ and

$$\left(c-\tfrac{1}{n}\right)^2 > c^2 - \frac{2c}{n} > 2,$$

putting $c - 1/n \in B$ below c. So the Cut Property fails over $\mathbf{Q}$.

Problem 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.
(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \le \sup B$.
(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution. (a) True. $\sup B$ is an upper bound for B, hence for the subset A, and $\sup A$ is the least upper bound for A (Definition 1.3.2 (ii)), so $\sup A \le \sup B$. Both suprema exist by the Axiom of Completeness, the sets being nonempty and bounded above.
(b) True. Take the midpoint $c = (\sup A + \inf B)/2$, which satisfies $\sup A < c < \inf B$. Then for all $a \in A$ and $b \in B$,

$$a \le \sup A < c < \inf B \le b.$$

(c) False. Let $A = (0,1)$, $B = (1,2)$, and $c = 1$: every $a \in A$ and $b \in B$ satisfies $a < 1 < b$, yet

$$\sup A = 1 = \inf B,$$

so the strict inequality fails.

Problem 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.
(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.
(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \ne 0$.
(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. (a) Write $a = p/q$ and $b = r/s$ with $p, r \in \mathbf{Z}$ and $q, s \in \mathbf{N}$. Then

$$a + b = \frac{ps + rq}{qs}, \qquad ab = \frac{pr}{qs},$$

and in each case the numerator lies in $\mathbf{Z}$ and the denominator qs lies in $\mathbf{N}$ (in particular $qs \ne 0$), because $\mathbf{Z}$ is closed under addition and multiplication. Hence $a + b, ab \in \mathbf{Q}$.
(b) Both are contrapositives of (a). If $a + t = r$ were rational, then $-a = (-1)a \in \mathbf{Q}$ and so

$$t = r + (-a) \in \mathbf{Q}$$

by (a), contradicting $t \in \mathbf{I}$. Likewise, if $a \ne 0$ and $at = r$ were rational, then $1/a \in \mathbf{Q}$ (if $a = p/q$ with $p \ne 0$, then $1/a = q/p$) and

$$t = \frac{1}{a} \cdot r \in \mathbf{Q}$$

by (a), again a contradiction.
(c) No, $\mathbf{I}$ is closed under neither operation, and in general nothing can be said: $s + t$ and st may be rational or irrational. With $s = \sqrt{2}$ (irrational by Theorem 1.1.1, and a real number by Theorem

1.4.5):

$$\begin{aligned} t = -\sqrt{2}: &\quad s + t = 0 \in \mathbf{Q}, \\ t = \sqrt{2}: &\quad s + t = 2\sqrt{2} \in \mathbf{I}, \quad st = 2 \in \mathbf{Q}, \\ t = 1 + \sqrt{2}: &\quad st = 2 + \sqrt{2} \in \mathbf{I}. \end{aligned}$$

(Part (b), applied to the rationals -1, 1 and 2, puts each of $-\sqrt{2}$, $1+\sqrt{2}$, $2\sqrt{2}$ and $2+\sqrt{2}$ in $\mathbf{I}$.)

Problem 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. s is an upper bound for A: if some $a \in A$ had $a > s$, then $a - s > 0$, and the Archimedean Property (Theorem 1.4.2 (ii)) supplies $n \in \mathbf{N}$ with

$$\frac{1}{n} < a - s, \qquad \text{i.e.} \qquad s + \frac{1}{n} < a,$$

contradicting the hypothesis that $s + \frac{1}{n}$ is an upper bound for A.
For leastness apply Lemma 1.3.8: let $\varepsilon > 0$ and choose (again by Theorem 1.4.2 (ii)) an $n \in \mathbf{N}$ with $1/n < \varepsilon$. Since $s - \frac{1}{n}$ is not an upper bound for A, there exists $a \in A$ with

$$s - \varepsilon < s - \frac{1}{n} < a.$$

Thus s is an upper bound admitting an element of A within every ε of it, so $s = \sup A$.

Problem 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution. Suppose $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. Then $x > 0$, so the Archimedean Property (Theorem 1.4.2 (ii)) applied to $y = x$ gives an $n_0 \in \mathbf{N}$ with

$$\frac{1}{n_0} < x,$$

whence $x \notin (0, 1/n_0)$ and x is not in the intersection after all. No such x exists, so the intersection is empty.
The sets $(0, 1/n)$ are nonempty, bounded and nested, $(0,1) \supseteq (0, 1/2) \supseteq \cdots$, so the only hypothesis of the Nested Interval Property (Theorem 1.4.1) that fails is closedness of the intervals.

Problem 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. b is an upper bound for T because $T \subseteq [a, b]$; the content is leastness, which is Lemma 1.3.8 fed by the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).
Let $\varepsilon > 0$ and set

$$c = \max\{a,\, b - \varepsilon\}.$$

Then $c < b$ (both $a < b$ and $b - \varepsilon < b$ hold), so Theorem 1.4.3 applied to the pair $c < b$ produces a rational r with $c < r < b$. From $c \geq a$ we get $a < r < b$, hence $r \in \mathbf{Q} \cap [a, b] = T$; from $c \geq b - \varepsilon$ we get

$$b - \varepsilon \leq c < r.$$

So every ε-neighborhood below b catches a point of T (in particular $T \neq \emptyset$), and Lemma 1.3.8 gives $\sup T = b$.

1.5 Exercises 1.4.5–1.5.3

Problem 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.
(Corollary 1.4.4: Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.)

Solution. Take $t = r + \sqrt{2}$, where $r \in \mathbf{Q}$ is chosen by the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) applied to the pair $a - \sqrt{2} < b - \sqrt{2}$, so that

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Adding $\sqrt{2}$ throughout gives $a < t < b$. And $t \in \mathbf{I}$: $\sqrt{2}$ is irrational by Theorem 1.1.1 (it exists as a real number by Theorem 1.4.5), so $r + \sqrt{2} \in \mathbf{I}$ by Exercise 1.4.1 (b) with the rational r.

Problem 1.4.6. Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.
(a) The set of all rational numbers p/q with $q \leq 10$.
(b) The set of all rational numbers p/q with q a power of 2.
(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution. Only (b) is dense.
(a) Not dense: the interval $(0, 1/11)$ contains no element. Indeed, if $p/q > 0$ with $p \in \mathbf{Z}$, $q \in \mathbf{N}$, $q \leq 10$, then $p \geq 1$ and

$$\frac{p}{q} \geq \frac{1}{q} \geq \frac{1}{10} > \frac{1}{11}.$$

(b) Dense. Let $a < b$. By the Archimedean Property (Theorem 1.4.2 (ii)) pick $n \in \mathbf{N}$ with $1/n < b - a$; since $2^n \geq n$ for all $n \in \mathbf{N}$ (induction: $2^1 = 2 \geq 1$, and $2^{n+1} = 2 \cdot 2^n \geq 2n \geq n + 1$), the denominator $q = 2^n$ satisfies

$$\frac{1}{2^n} \leq \frac{1}{n} < b - a.$$

Now run the argument of Theorem 1.4.3 with this q: let $p \in \mathbf{Z}$ be the smallest integer greater than $2^n a$, so that

$$p - 1 \leq 2^n a < p.$$

The right inequality gives $a < p/2^n$, and the left one gives

$$\begin{aligned} p &\leq 2^n a + 1 \\ &< 2^n a + 2^n(b - a) \\ &= 2^n b, \end{aligned}$$

so $p/2^n < b$. Hence $a < p/2^n < b$ with $p/2^n$ in the set.
(c) Not dense: the interval $(0, 1/10)$ contains no element, since $10|p| \geq q$ forces

$$\left|\frac{p}{q}\right| = \frac{|p|}{q} \geq \frac{1}{10}.$$

(Note $p = 0$ is excluded outright, as $10|p| \geq q \geq 1$ fails.)

Problem 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.
(Theorem 1.4.5: There exists a real number $\alpha \in \mathbf{R}$ satisfying $\alpha^2 = 2$. In the proof, $T = \{t \in$

$\mathbf{R} : t^2 < 2\}$ and $\alpha = \sup T$; the case $\alpha^2 < 2$ has already been ruled out, and the computation $(\alpha - 1/n)^2 > \alpha^2 - 2\alpha/n$ has been established.)

Solution. Assume $\alpha^2 > 2$. Note first $\alpha \geq 1 > 0$, since $1 \in T$ and α is an upper bound for T, so $(\alpha^2 - 2)/(2\alpha) > 0$ and the Archimedean Property (Theorem 1.4.2 (ii)) provides $n_0 \in \mathbf{N}$ with

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}, \qquad \text{i.e.} \qquad \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

Feeding this into the displayed estimate of the proof,

$$\begin{aligned}\left(\alpha - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{n_0} \\ &> \alpha^2 - (\alpha^2 - 2) \\ &= 2.\end{aligned}$$

Also $\alpha - 1/n_0 > 0$, because $1/n_0 < (\alpha^2 - 2)/(2\alpha) < \alpha^2/(2\alpha) = \alpha/2$.
Now $\alpha - 1/n_0$ is an upper bound for T: given $t \in T$, either $t \leq 0 < \alpha - 1/n_0$, or $t > 0$ and then

$$t^2 < 2 < \left(\alpha - \frac{1}{n_0}\right)^2$$

forces $t < \alpha - 1/n_0$ (for positive reals x, y, $x \geq y$ implies $x^2 \geq y^2$). So T has an upper bound strictly smaller than α, contradicting the leastness half of $\alpha = \sup T$. Hence $\alpha^2 > 2$ is impossible, and with $\alpha^2 < 2$ already excluded, $\alpha^2 = 2$.

Problem 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.
(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.
(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)
(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution. (a), (b), (c) are possible; (d) is impossible.
(a) Take

$$A = \left\{1 - \tfrac{1}{2n} : n \in \mathbf{N}\right\}, \qquad B = \left\{1 - \tfrac{1}{2n-1} : n \in \mathbf{N}\right\}.$$

Since $m \mapsto 1 - 1/m$ is injective and no integer is both even and odd, $A \cap B = \emptyset$. Both sets are bounded above by 1 and miss it. Given $c < 1$, the Archimedean Property (Theorem 1.4.2) supplies m with $1/m < 1 - c$, hence also $1/(m+1) < 1 - c$; one of $m, m+1$ is even and the other odd, so each of A, B contains a point exceeding c. Thus $\sup A = \sup B = 1 \notin A \cup B$.
(b) Take $J_n = (-1/n, 1/n)$, which is nested since $1/(n+1) < 1/n$. Then

$$\bigcap_{n=1}^{\infty} J_n = \{0\},$$

a single element: $0 \in J_n$ for all n, while for $x \neq 0$ the Archimedean Property supplies n with $1/n < |x|$, so $x \notin J_n$.
(c) Take $L_n = [n, \infty)$, nested because $n < n + 1$. Given $x \in \mathbf{R}$, the Archimedean Property supplies $n \in \mathbf{N}$ with $n > x$, so $x \notin L_n$; hence $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

(d) Impossible. Write $I_n = [a_n, b_n]$ and set $A_N = \bigcap_{n=1}^{N} I_n$, so

$$A_N = \Big[\max_{1 \le n \le N} a_n,\ \min_{1 \le n \le N} b_n \Big]$$

is a closed bounded interval, nonempty by hypothesis, and $A_{N+1} = A_N \cap I_{N+1}$ makes the A_N nested. So the Nested Interval Property (Theorem 1.4.1) applies to (A_N), giving

$$\begin{aligned} \bigcap_{n=1}^{\infty} I_n &= \bigcap_{N=1}^{\infty} \bigcap_{n=1}^{N} I_n \\ &= \bigcap_{N=1}^{\infty} A_N \neq \emptyset. \end{aligned}$$

Problem 1.5.1. Finish the following proof for Theorem 1.5.7.
Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is 1–1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.
Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1–1 function g from $\mathbf{N}$ onto A.

Solution. Having chosen $n_1 < n_2 < \cdots < n_k$, set

$$n_{k+1} \ = \ \min\{\, n \in \mathbf{N} : n > n_k \text{ and } f(n) \in A \,\}, \qquad g(k+1) = f(n_{k+1}).$$

The set being minimized is nonempty: f is 1–1 and onto $B \supseteq A$, so f restricts to a bijection of $S = \{n : f(n) \in A\}$ onto A, whence S is infinite and meets $\{n : n > n_k\}$. The minimum then exists by the Well-Ordering Property of $\mathbf{N}$, so $g : \mathbf{N} \to A$ is defined on all of $\mathbf{N}$, and $g(k) = f(n_k) \in A$ for every k.
g is 1–1. The sequence (n_k) is strictly increasing by construction, so $j \neq k$ gives $n_j \neq n_k$, and f being 1–1 gives $g(j) = f(n_j) \neq f(n_k) = g(k)$.
g is onto A. Fix $a \in A$ and let $m = f^{-1}(a)$, so $m \in S$. Since $n_k \ge k$ for all k (a strictly increasing sequence in $\mathbf{N}$), the set $\{k : n_k \ge m\}$ is nonempty; let k be its least element. Two cases:
(i) $k = 1$. Then $n_1 = \min S \le m$ because $m \in S$, and $n_1 \ge m$, so $n_1 = m$.
(ii) $k > 1$. Then $n_{k-1} < m \le n_k$ by minimality of k, so m is a competitor in the minimum defining n_k; hence $n_k \le m$, and therefore $n_k = m$.
In both cases $g(k) = f(m) = a$. Thus g is a 1–1 correspondence between $\mathbf{N}$ and A, i.e. $\mathbf{N} \sim A$, so A is countable by Definition 1.5.5.

Problem 1.5.2. Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:
Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution. The step "our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$" is false. What the construction actually gives is

$$\Big(\bigcap_{n=1}^{\infty} I_n \Big) \cap \mathbf{Q} = \emptyset,$$

since the only points excluded are the r_n, and those exhaust $\mathbf{Q}$, not $\mathbf{R}$.
In Theorem 1.5.6 (ii) the list $\{x_1, x_2, x_3, \ldots\}$ was assumed to be all of $\mathbf{R}$, so ruling out every listed point ruled out every point whatsoever and forced the intersection to be empty. Here the intervals I_n are still intervals of *real* numbers, and NIP (Theorem 1.4.1) supplies some $x \in \bigcap_{n=1}^{\infty} I_n$. The displayed line says only that this x is irrational — a true and unsurprising statement, consistent with $\mathbf{Q}$ being

countable (Theorem 1.5.6 (i)). No contradiction arises.

Problem 1.5.3. Use the following outline to supply proofs for the statements in Theorem 1.5.8.
(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)
Now, explain how the more general statement in (i) follows.
(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part (i).
(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

Solution. (a) Interleave the two lists. Let $B_2 = A_2 \setminus A_1$, so $A_1 \cup A_2 = A_1 \cup B_2$ with $A_1 \cap B_2 = \emptyset$. Since $B_2 \subseteq A_2$ and A_2 is countable, Theorem 1.5.7 says B_2 is countable or finite.
(i) B_2 countable. Take bijections $f : \mathbf{N} \to A_1$ and $g : \mathbf{N} \to B_2$ and define

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \end{cases}$$

exactly the odd/even split used for $\mathbf{N} \sim \mathbf{Z}$ in Example 1.5.3 (ii). Then h is onto because f and g are, and h is 1–1 because f and g are and their ranges A_1, B_2 are disjoint.
(ii) $B_2 = \{b_1, \ldots, b_k\}$ finite (possibly empty). Put $h(n) = b_n$ for $1 \le n \le k$ and $h(n) = f(n-k)$ for $n > k$; the same two clauses certify 1–1 and onto.
Either way $A_1 \cup A_2 \sim \mathbf{N}$. For general m, induct: $A_1 \cup \cdots \cup A_{m+1} = (A_1 \cup \cdots \cup A_m) \cup A_{m+1}$ is the union of a countable set (inductive hypothesis) with a countable set, hence countable by the two-set case.
(b) Induction proves a statement $P(m)$ for every $m \in \mathbf{N}$, and here $P(m)$ reads "$A_1 \cup \cdots \cup A_m$ is countable" — a statement about a *finite* union. The set $\bigcup_{n=1}^{\infty} A_n$ is not $A_1 \cup \cdots \cup A_m$ for any m, so it is never the subject of any $P(m)$; there is no final "$m = \infty$" step to take. The gap is genuine, not cosmetic: with $A_n = \{n\}$, every $A_1 \cup \cdots \cup A_m$ is finite, yet $\bigcup_{n=1}^{\infty} A_n = \mathbf{N}$ is not.
(c) The array exhibits a 1–1, onto map $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$; counting along reverse diagonals, the entry in row n, column m is

$$\sigma(n,m) \;=\; \frac{(n+m-1)(n+m)}{2} \;-\; (n-1).$$

Each diagonal $n+m-1 = d$ contributes the d consecutive integers from $\frac{d(d-1)}{2}+1$ to $\frac{d(d+1)}{2}$, read bottom-to-top, and these blocks tile $\mathbf{N}$ without overlap — so σ is a bijection. (Check: $\sigma(1,1) = 1$, $\sigma(2,1) = 2$, $\sigma(1,2) = 3$, $\sigma(3,1) = 4$, $\sigma(2,2) = 5$.)
Now let A_n be countable for each $n \in \mathbf{N}$ and write $A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$ using a bijection $\mathbf{N} \to A_n$. Put $A = \bigcup_{n=1}^{\infty} A_n$ and define $\varphi : A \to \mathbf{N}$ by

$$\varphi(x) = \sigma(n,m), \qquad \begin{array}{l} n = \min\{k : x \in A_k\}, \\ m = \text{the index of } x \text{ in the list for } A_n. \end{array}$$

Both n and m are uniquely determined by x (the first by the Well-Ordering Property, the second because the list for A_n repeats nothing), so φ is well defined; and it is 1–1 because σ is injective and the pair (n,m) recovers x as a_{nm}. Hence $A \sim \varphi(A) \subseteq \mathbf{N}$, and Theorem 1.5.7 makes $\varphi(A)$ — so also A — countable or finite. Since $A \supseteq A_1$ is infinite, A is countable.

1.6 Exercises 1.5.4–1.5.10

Problem 1.5.4. (a) Show $(a,b) \sim \mathbf{R}$ for any interval (a,b).
(b) Show that an unbounded interval like $(a,\infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.
(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necessary. Show that $[0,1) \sim (0,1)$ by exhibiting a 1–1 onto function between the two sets.

Solution. (a) Straighten (a,b) onto $(-1,1)$ and quote Example 1.5.4. The affine map

$$h(x) = \frac{2x-a-b}{b-a}$$

is strictly increasing with inverse $h^{-1}(t) = \frac{1}{2}\big((b-a)t + a + b\big)$, and $h(a) = -1$, $h(b) = 1$, so h carries (a,b) 1–1 onto $(-1,1)$. Example 1.5.4 gives $(-1,1) \sim \mathbf{R}$ via $f(x) = x/(x^2-1)$, so $x \mapsto f(h(x))$ is a 1–1 correspondence $(a,b) \to \mathbf{R}$ by the transitivity of $\sim$ (Exercise 1.5.5 (c)).
(b) Take

$$k(x) = \frac{1}{x-a+1}, \qquad x \in (a,\infty).$$

For $x > a$ we have $x-a+1 > 1$, so $k(x) \in (0,1)$; and k is 1–1 onto $(0,1)$ because $y \mapsto a-1+1/y$ inverts it, mapping $(0,1)$ back into (a,∞) (indeed $1/y > 1$ exactly when $0 < y < 1$). Thus $(a,\infty) \sim (0,1) \sim \mathbf{R}$ by (a) and transitivity.
(c) Shift a sequence one place along (Hilbert's hotel). Let $D = \{0\} \cup \{1/n : n \geq 2\}$ and define $g : [0,1) \to (0,1)$ by

$$g(x) = \begin{cases} 1/2 & \text{if } x = 0, \\ 1/(n+1) & \text{if } x = 1/n,\ n \geq 2, \\ x & \text{if } x \in [0,1) \setminus D. \end{cases}$$

Then $g(D) = \{1/n : n \geq 2\}$ is a 1–1 image of D (the map $0 \mapsto 1/2$, $1/n \mapsto 1/(n+1)$ is injective), g is the identity on $[0,1) \setminus D = (0,1) \setminus \{1/n : n \geq 2\}$, and these two image sets are disjoint and have union $(0,1)$. Hence g is 1–1 and onto, and $[0,1) \sim (0,1)$.

Problem 1.5.5. (a) Why is $A \sim A$ for every set A?
(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.
(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an equivalence relation.

Solution. (a) The identity $i : A \to A$, $i(x) = x$, is 1–1 ($i(x) = i(y)$ says $x = y$) and onto ($x = i(x)$), so $A \sim A$ by Definition 1.5.2.
(b) If $f : A \to B$ is 1–1 and onto, then each $y \in B$ has exactly one preimage — at least one by ontoness, at most one by injectivity — so $f^{-1} : B \to A$, $f^{-1}(y) =$ that preimage, is a function. It is onto (given $x \in A$, $f^{-1}(f(x)) = x$) and 1–1 (if $f^{-1}(y_1) = f^{-1}(y_2) = x$ then $y_1 = f(x) = y_2$). Hence $A \sim B$ gives $B \sim A$, and interchanging the roles of A and B gives the converse.
(c) Compose. Let $f : A \to B$ and $g : B \to C$ both be 1–1 and onto, and set $h = g \circ f : A \to C$. Then

$$\begin{aligned} h(x_1) = h(x_2) \ &\Rightarrow\ f(x_1) = f(x_2) \quad && (g \text{ is 1–1}) \\ &\Rightarrow\ x_1 = x_2 && (f \text{ is 1–1}), \end{aligned}$$

so h is 1–1. Given $z \in C$, ontoness of g supplies $y \in B$ with $g(y) = z$, and ontoness of f supplies $x \in A$ with $f(x) = y$; then $h(x) = g(f(x)) = z$, so h is onto. Thus $A \sim C$.

Problem 1.5.6. (a) Give an example of a countable collection of disjoint open intervals.
(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution. (a) $\{(n, n+1) : n \in \mathbf{Z}\}$. Distinct integers $m < n$ give $(m, m+1) \cap (n, n+1) = \emptyset$ since $m + 1 \leq n$, and the collection is countable because $n \mapsto (n, n+1)$ is a 1–1 correspondence with $\mathbf{Z}$, which is countable by Example 1.5.3 (ii) and Exercise 1.5.5 (c).
(b) No such collection exists: every collection of pairwise disjoint (nonempty) open intervals is countable or finite. Let $\mathcal{I}$ be such a collection. By the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), each $I = (a, b) \in \mathcal{I}$ contains a rational; choose one and call it q_I. The map

$$\mathcal{I} \longrightarrow \mathbf{Q}, \qquad I \longmapsto q_I,$$

is 1–1: if $q_I = q_J$ then that rational lies in $I \cap J$, so $I \cap J \neq \emptyset$ and disjointness forces $I = J$. Hence $\mathcal{I} \sim \{q_I : I \in \mathcal{I}\} \subseteq \mathbf{Q}$, and since $\mathbf{Q}$ is countable (Theorem 1.5.6 (i)), Theorem 1.5.7 makes this subset — and therefore $\mathcal{I}$ — countable or finite.

Problem 1.5.7. Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.
(a) Find a 1–1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)
(b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . ..)
The Schroeder–Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution. (a) $f(x) = (x, 1/2)$. It lands in S because $0 < x < 1$ and $0 < 1/2 < 1$, and $f(x_1) = f(x_2)$ forces $x_1 = x_2$ on the first coordinate. (It is far from onto: its range is a single horizontal segment.)
(b) Interleave the digits. Fix once and for all the *non-terminating* decimal expansion of each point of $(0, 1)$: every $t \in (0, 1)$ has exactly one expansion $t = .t_1t_2t_3 \ldots$ whose digits are not eventually all 0 (so 0.235 is written .234999 . . .). With $x = .a_1a_2a_3 \ldots$ and $y = .b_1b_2b_3 \ldots$ in that form, define

$$g(x, y) \;=\; .\,a_1b_1a_2b_2a_3b_3 \ldots$$

Range. Since $x > 0$, some $a_i \neq 0$, so $g(x, y) > 0$; since $x < 1$ and x is not .999 . . ., some $a_i \neq 9$, so $g(x, y) < 1$. Thus $g : S \to (0, 1)$.
1–1. The interleaved digit string never ends in all 0's, because that would force both (a_i) and (b_i) to end in all 0's, which the convention forbids. Two distinct decimal strings represent the same real only when one ends in all 0's and the other in all 9's; hence distinct interleavings give distinct reals. If $(x, y) \neq (x', y')$ then $(a_i) \neq (a_i')$ or $(b_i) \neq (b_i')$ — expansions being unique in our convention — so the interleavings differ as strings and therefore as numbers. Hence g is 1–1.
Onto? No. The number $z = .1505050 \ldots$ neither terminates nor ends in all 9's, so this is its only decimal string; de-interleaving it forces $a = .1000 \ldots$, a terminating string, which our convention assigns to no $x \in (0, 1)$. Hence $z \notin g(S)$.
With f and g in hand, the Schroeder–Bernstein Theorem (Exercise 1.5.11) yields $(0, 1) \sim S$.

Problem 1.5.8. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution. Slice B by size: for each $n \in \mathbf{N}$ put

$$B_n = \{\, b \in B : b \geq 1/n \,\}.$$

Each B_n is finite. Indeed, if $b_1, \ldots, b_k$ are distinct elements of B_n, then the hypothesis applied to this finite subset gives

$$2 \;\geq\; b_1 + b_2 + \cdots + b_k \;\geq\; \frac{k}{n},$$

so $k \leq 2n$; thus B_n has at most $2n$ elements.
Every $b \in B$ satisfies $b > 0$, so the Archimedean Property (Theorem 1.4.2) supplies an n with $1/n < b$, i.e. $b \in B_n$. Hence

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Theorem 1.5.8 (ii) is stated for countable sets, so pad: each $B_n \cup \mathbf{N}$ is countable, since listing the finitely many points of $B_n \setminus \mathbf{N}$ and then $1, 2, 3, \ldots$ enumerates it. Hence

$$\bigcup_{n=1}^{\infty} (B_n \cup \mathbf{N}) \;=\; B \cup \mathbf{N}$$

countable. Since $B \subseteq B \cup \mathbf{N}$, Theorem 1.5.7 gives that B is countable or finite.

Problem 1.5.9. A real number $x \in \mathbf{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.
(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.
(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.
(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution. (a) Exhibit the polynomials: $\sqrt{2}$ is a root of $x^2 - 2$, and $\sqrt[3]{2}$ is a root of $x^3 - 2$. For the third, square twice:

$$\begin{aligned} x = \sqrt{3} + \sqrt{2} \;&\Rightarrow\; x^2 = 5 + 2\sqrt{6} \\ &\Rightarrow\; (x^2 - 5)^2 = 24 \\ &\Rightarrow\; x^4 - 10x^2 + 1 = 0, \end{aligned}$$

so $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$.
(b) First, $\mathbf{Z}^k$ is countable for every $k \in \mathbf{N}$, by induction: $\mathbf{Z}$ is countable (Example 1.5.3 (ii)), and writing $\mathbf{Z}$ as a sequence $m_1, m_2, m_3, \ldots$ gives

$$\mathbf{Z}^{k+1} = \bigcup_{j=1}^{\infty} \{m_j\} \times \mathbf{Z}^k,$$

each slice a copy of $\mathbf{Z}^k$: a countable union of countable sets, hence countable by Theorem 1.5.8 (ii).
Now let P_n be the set of degree-n polynomials with integer coefficients. Sending $p(x) = a_n x^n + \cdots + a_0$ to $(a_0, \ldots, a_n)$ is a 1–1 map of P_n into $\mathbf{Z}^{n+1}$, so P_n is countable or finite by Theorem 1.5.7; it is infinite, hence countable. Enumerate $P_n = \{p_1, p_2, p_3, \ldots\}$ and let $R_j \subseteq \mathbf{R}$ be the root set of p_j, which is finite (at most n elements). Padding as in Exercise 1.5.8, each $R_j \cup \mathbf{N}$ is countable, so

$$\bigcup_{j=1}^{\infty} (R_j \cup \mathbf{N}) \;=\; A_n \cup \mathbf{N}$$

is countable by Theorem 1.5.8 (ii), and A_n is countable or finite by Theorem 1.5.7. It is infinite: each $m \in \mathbf{Z}$ is a root of $x^n - mx^{n-1} \in P_n$. So A_n is countable.
(c) The set of algebraic numbers is $A = \bigcup_{n=1}^{\infty} A_n$, countable by Theorem 1.5.8 (ii). If the set $T = \mathbf{R} \setminus A$ of transcendental numbers were countable or finite, then $\mathbf{R} = A \cup T$ would be countable (Theorem 1.5.8 (i), after padding a finite T with $\mathbf{N}$), contradicting Theorem 1.5.6 (ii). Hence T is uncountable; in particular transcendental numbers exist.

Problem 1.5.10. (a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.
(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?
(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution. (a) Some $a = 1/n$ works. Suppose instead that $C \cap [1/n, 1]$ is countable or finite for every $n \geq 2$. Every $x \in C$ with $x > 0$ satisfies $x \geq 1/n$ for some such n (Theorem 1.4.2), so

$$C \subseteq \{0\} \cup \bigcup_{n=2}^{\infty} \bigl(C \cap [1/n, 1]\bigr).$$

Padding each piece with $\mathbf{N}$ as in Exercise 1.5.8, the right-hand side is countable by Theorem 1.5.8 (ii), whence C is countable or finite by Theorem 1.5.7 — contradicting that C is uncountable.
(b) No — and not just occasionally: $C \cap [\alpha, 1]$ is never uncountable. By (a) the set A is nonempty, and it is downward closed in $(0,1)$, since $0 < a' < a$ gives $C \cap [a', 1] \supseteq C \cap [a, 1]$. Thus $\alpha = \sup A \in (0, 1]$, and there are two cases.
(i) $\alpha = 1$. Then $C \cap [\alpha, 1] = C \cap \{1\}$ has at most one element.
(ii) $\alpha < 1$. Every a with $\alpha < a < 1$ exceeds $\sup A$, so $a \notin A$ and $C \cap [a, 1]$ is countable or finite. Fix N with $1/N < 1 - \alpha$; each $x \in C$ with $x > \alpha$ satisfies $x \geq \alpha + 1/n$ for some $n \geq N$, so

$$C \cap [\alpha, 1] \subseteq \{\alpha\} \cup \bigcup_{n=N}^{\infty} \bigl(C \cap [\alpha + 1/n, 1]\bigr),$$

again countable or finite by Theorem 1.5.8 (ii) and Theorem 1.5.7.
(c) No. Take $C = \{1/n : n \in \mathbf{N}\} \subseteq [0,1]$, which is infinite. For any $a \in (0,1)$,

$$C \cap [a, 1] = \{\, 1/n : n \leq 1/a \,\}$$

is finite.

1.7 Exercises 1.5.11–1.6.6

Problem 1.5.11. (Schröder–Bernstein Theorem). Assume there exists a 1–1 function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Follow the steps to show that there exists a 1–1, onto function $h : X \to Y$ and hence $X \sim Y$.
The strategy is to partition X and Y into components

$$X = A \cup A' \qquad \text{and} \qquad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.
(a) Explain how achieving this would lead to a proof that $X \sim Y$.
(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.
(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.
(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution. (a) Glue the two bijections. Since g is 1–1 and maps B' onto A', its restriction to B' is a bijection $B' \to A'$, so $g^{-1}(x) \in B'$ is unambiguous for $x \in A'$. Define

$$h(x) = \begin{cases} f(x), & x \in A, \\ g^{-1}(x), & x \in A'. \end{cases}$$

This is defined on all of $X = A \cup A'$ and unambiguous because $A \cap A' = \emptyset$. It is onto: $h(A) = f(A) = B$ and $h(A') = B'$, and $B \cup B' = Y$. It is 1–1: f is 1–1 on A, g^{-1} is 1–1 on A', and the two images B and B' are disjoint, so no collision can occur across the cases. Thus $X \sim Y$ by Definition 1.5.2.
(b) If $A_1 = \emptyset$ then $g(Y) = X$, so g is itself 1–1 and onto and $X \sim Y$ at once (Exercise 1.5.5 (b)); the construction below degenerates harmlessly to $A = B = \emptyset$, $A' = X$, $B' = Y$, $h = g^{-1}$. In general, set $\varphi = g \circ f : X \to X$, which is 1–1 as a composition of 1–1 maps. Then $A_{n+1} = \varphi(A_n)$, so

$$A_n = \varphi^{n-1}(A_1) \qquad (n \in \mathbf{N}).$$

Two observations. First, a 1–1 map ψ satisfies $\psi(S) \cap \psi(T) = \psi(S \cap T)$ for all sets S, T (if $\psi(s) = \psi(t)$ then $s = t$). Second, for $n \geq 2$ we have $A_n = g(f(A_{n-1})) \subseteq g(Y)$, whereas $A_1 \cap g(Y) = \emptyset$; hence

$$A_1 \cap A_n = \emptyset \qquad (n \geq 2).$$

Now take $1 \leq m < n$ and put $k = n - m + 1 \geq 2$. Applying the first observation to the 1–1 map φ^{m-1},

$$\begin{aligned} A_m \cap A_n &= \varphi^{m-1}(A_1) \cap \varphi^{m-1}(A_k) \\ &= \varphi^{m-1}(A_1 \cap A_k) = \emptyset. \end{aligned}$$

So $\{A_n\}$ is pairwise disjoint, and since f is 1–1,

$$f(A_m) \cap f(A_n) = f(A_m \cap A_n) = \emptyset \qquad (m \neq n),$$

giving the same for $\{f(A_n)\}$ in Y.
(c) Images carry unions to unions:

$$f(A) = f\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} f(A_n) = B,$$

so f maps A onto B; being a restriction of a 1–1 map, it does so 1–1.
(d) Both inclusions use injectivity of g.
(i) $g(B') \subseteq A'$. Let $y \in B'$ and suppose $g(y) \in A_n$ for some n. Since $g(y) \in g(Y)$ and $A_1 \cap g(Y) = \emptyset$, necessarily $n \geq 2$, so $g(y) \in g(f(A_{n-1}))$, say $g(y) = g(y')$ with $y' \in f(A_{n-1})$. As g is 1–1, $y = y' \in f(A_{n-1}) \subseteq B$, contradicting $y \in B'$. Hence $g(y) \notin A$, i.e. $g(y) \in A'$.
(ii) $A' \subseteq g(B')$. Let $x \in A'$. Then $x \notin A_1 = X \setminus g(Y)$, so $x = g(y)$ for some $y \in Y$. If $y \in B$, then $y \in f(A_n)$ for some n and therefore

$$x = g(y) \in g(f(A_n)) = A_{n+1} \subseteq A,$$

contradicting $x \in A'$. So $y \in B'$ and $x \in g(B')$.
Thus g maps B' onto A', and with (c) the hypotheses of (a) hold, delivering h.

Problem 1.6.1. Show that $(0,1)$ is uncountable if and only if $\mathbf{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

Solution. Both sets are infinite and $(0,1) \sim \mathbf{R}$, so neither is countable without the other.
By Example 1.5.4 the map $x \mapsto x/(x^2-1)$ carries $(-1,1)$ onto $\mathbf{R}$ in a 1–1 fashion, and $x \mapsto 2x-1$ carries $(0,1)$ onto $(-1,1)$ in a 1–1 fashion, so transitivity (Exercise 1.5.5(c)) gives

$$(0,1) \sim (-1,1) \sim \mathbf{R}.$$

Hence $\mathbf{N} \sim (0,1)$ implies $\mathbf{N} \sim \mathbf{R}$, and $\mathbf{N} \sim \mathbf{R}$ implies $\mathbf{N} \sim (0,1)$ (using symmetry, Exercise 1.5.5(b)): one set is countable exactly when the other is. Both are infinite, each containing $\{1/n : n \geq 2\}$, so Definition 1.5.5 makes "uncountable" mean "not countable" for each, and

$$(0,1) \text{ uncountable} \iff \mathbf{R} \text{ uncountable}. \qquad \blacksquare$$

Problem 1.6.2. In the proof of Theorem 1.6.1 one assumes, for contradiction, that $f : \mathbf{N} \to (0,1)$ is 1–1 and onto, and writes each value in decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots,$$

so that $a_{mn} \in \{0,1,2,\ldots,9\}$ is the nth digit in the decimal expansion of $f(m)$. The 1–1 correspondence between $\mathbf{N}$ and $(0,1)$ is then summarised by the doubly indexed array

$$\begin{array}{lcl} 1 & \longleftrightarrow & f(1) = .a_{11}\,a_{12}\,a_{13}\,a_{14}\cdots \\ 2 & \longleftrightarrow & f(2) = .a_{21}\,a_{22}\,a_{23}\,a_{24}\cdots \\ 3 & \longleftrightarrow & f(3) = .a_{31}\,a_{32}\,a_{33}\,a_{34}\cdots \\ & \vdots & \end{array}$$

the key assumption being that every real number in $(0,1)$ appears somewhere on the list. Define a real number $x \in (0,1)$ with decimal expansion $x = .b_1b_2b_3b_4\ldots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2, \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

(a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.
(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.
(c) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution. (a) Because $b_1 \neq a_{11}$, and the expansion of x is the only one x has.
The rule gives $b_1 = 3$ when $a_{11} = 2$ and $b_1 = 2$ when $a_{11} \neq 2$; in both cases $b_1 \neq a_{11}$. Every digit of x lies in $\{2,3\}$, so the expansion $.b_1b_2b_3\ldots$ neither terminates nor ends in repeating 9's, and is therefore the unique decimal representation of x (this is the point taken up in Exercise 1.6.3(b)). Hence if $x = f(1)$, the expansion $.a_{11}a_{12}a_{13}\ldots$ would have to be that unique representation, forcing $a_{11} = b_1$ – false.
(b) Identical, one row down: $b_2 \neq a_{22}$ by the rule, so x and $f(2)$ disagree in the second decimal digit and $x \neq f(2)$. In general, for every $n \in \mathbf{N}$,

$$b_n \neq a_{nn} \quad\Longrightarrow\quad x \neq f(n),$$

since equality would force the unique expansion of x to be $.a_{n1}a_{n2}a_{n3}\ldots$, whose nth digit is $a_{nn} \neq b_n$.
(c) The number x satisfies

$$\tfrac{2}{9} = .222\ldots \;\leq\; x \;\leq\; .333\ldots = \tfrac{1}{3},$$

so $x \in (0,1)$. But by (b), $x \neq f(n)$ for every $n \in \mathbf{N}$, so x is not in the range of f – contradicting the assumption that f is onto. No 1–1, onto $f : \mathbf{N} \to (0,1)$ exists, so $\mathbf{N} \not\sim (0,1)$; since $(0,1)$ is infinite (it contains $\{1/n : n \geq 2\}$), Definition 1.5.5 declares $(0,1)$ uncountable. ∎

Problem 1.6.3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.
(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.
(b) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\ldots$. Doesn't this cause some problems?

Solution. (a) Run the argument on $\mathbf{Q} \cap (0,1)$ and nothing breaks: the diagonal number x escapes the list, which says only that x is irrational.
Let $f : \mathbf{N} \to \mathbf{Q} \cap (0,1)$ be 1–1 and onto; such an f exists because $\mathbf{Q} \cap (0,1)$ is an infinite subset (it contains every $1/n$, $n \geq 2$) of the countable set $\mathbf{Q}$, so Theorem 1.5.6(i) and Theorem 1.5.7 make it countable. The construction still produces a real $x \in (0,1)$ differing from $f(n)$ in the nth digit for every n, so x lies outside the range of f. But that range is $\mathbf{Q} \cap (0,1)$, not $(0,1)$, so no contradiction follows – only the true statement

$$x \in (0,1) \setminus \mathbf{Q}.$$

Theorem 1.6.1 gets its contradiction because there f is assumed onto $(0,1)$, a set that visibly contains x.
(b) It causes none, because the digits were chosen from $\{2,3\}$ exactly to avoid it.
Two conventions settle the matter. First, for each m fix one decimal representation of $f(m)$ (say, the one not ending in repeating 9's), so that the array entries a_{mn} are well defined. Second, observe that the constructed $x = .b_1b_2b_3\ldots$ has every digit equal to 2 or 3: its expansion neither terminates nor ends in repeating 9's, so it is the only decimal representation x has. Hence

$$x = f(m) \implies .a_{m1}a_{m2}a_{m3}\ldots = .b_1b_2b_3\ldots \implies a_{mm} = b_m,$$

contradicting $b_m \neq a_{mm}$. ∎

Problem 1.6.4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S, as is the sequence $(1,1,1,1,1,1,\ldots)$. Give a rigorous argument showing that S is uncountable.

Solution. Diagonalize, flipping bits: the escaping sequence is $s = (1 - a_{11},\, 1 - a_{22},\, 1 - a_{33},\, \ldots)$.
First, S is infinite, since $n \mapsto e_n$, where e_n has a 1 in the nth slot and 0's elsewhere, is a 1–1 map of $\mathbf{N}$ into S.
Now assume, for contradiction, that S is countable, so that there is a 1–1, onto $f : \mathbf{N} \to S$. Write the mth listed sequence as

$$f(m) = (a_{m1},\, a_{m2},\, a_{m3},\, \ldots), \qquad a_{mn} \in \{0,1\},$$

and set $b_n = 1 - a_{nn}$, $s = (b_1, b_2, b_3, \ldots)$. Each b_n is 0 or 1, so $s \in S$. For every $n \in \mathbf{N}$ the nth terms satisfy

$$b_n = 1 - a_{nn} \neq a_{nn},$$

and two sequences are equal precisely when all of their corresponding terms agree; hence $s \neq f(n)$ for every n. Thus $s \in S$ is not in the range of f, contradicting the assumption that f is onto.
Therefore $\mathbf{N} \not\sim S$, and since S is infinite, S is uncountable by Definition 1.5.5. ∎

Problem 1.6.5. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution. (a) The eight subsets are

$$P(A) = \{\, \emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\} \,\}.$$

(b) Induction on n, splitting the subsets according to whether they contain a fixed element.
For $n = 0$ we have $A = \emptyset$ and $P(\emptyset) = \{\emptyset\}$, which has $1 = 2^0$ element. Assume every n-element set has a power set with 2^n elements, let A have $n+1$ elements, fix $x \in A$, and put $A' = A \setminus \{x\}$, a set with n elements. Every subset of A either omits x or contains it, so

$$P(A) = P(A') \ \cup\ \{\, E \cup \{x\} : E \in P(A') \,\},$$

and the two pieces are disjoint. The first has 2^n elements by the induction hypothesis, and $E \mapsto E \cup \{x\}$ maps $P(A')$ onto the second in a 1–1 fashion (inverse $F \mapsto F \setminus \{x\}$), so the second has 2^n elements too. Hence $P(A)$ has $2^n + 2^n = 2^{n+1}$ elements. ∎

Problem 1.6.6. (a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1–1 mappings from A into $P(A)$.
(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1–1 map $g : C \to P(C)$.
(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.

Solution. (a) The singleton map and a nested chain:

$$\begin{aligned} f_1 &: a \mapsto \{a\}, \quad b \mapsto \{b\}, \quad c \mapsto \{c\}; \\ f_2 &: a \mapsto \emptyset, \qquad b \mapsto \{a\}, \quad c \mapsto \{a,b\}. \end{aligned}$$

Each lists three distinct subsets of A, so each is 1–1.
(b) $g(k) = \{k\}$, that is

$$g(1) = \{1\}, \quad g(2) = \{2\}, \quad g(3) = \{3\}, \quad g(4) = \{4\},$$

four distinct elements of $P(C)$, so g is 1–1.
(c) Counting: a map defined on A has at most three values while $P(A)$ has eight, and one defined on C has at most four while $P(C)$ has sixteen.
The range $\{f(a) : a \in A\}$ acquires at most one element per element of A, so it has at most three members; by Exercise 1.6.5(b), $P(A)$ has $2^3 = 8$ members and $P(C)$ has $2^4 = 16$. Since $3 < 8$ and $4 < 16$, the range is in each case a proper subset of the power set, so no such map is onto. ∎

1.8 Exercises 1.6.7–1.6.10

Problem 1.6.7. Return to the particular functions constructed in Exercise 1.6.6 and construct the subset

$$B = \{a \in A : a \notin f(a)\}$$

that results using the preceding rule. In each case, note that B is not in the range of the function used.

Solution. $B = \emptyset$ for both singleton maps, and $B = A$ for the nested chain.
(i) $f_1 : A \to P(A)$, $f_1(x) = \{x\}$, on $A = \{a, b, c\}$. Every x satisfies $x \in \{x\} = f_1(x)$, so no element qualifies:

$$B = \{x \in A : x \notin \{x\}\} = \emptyset,$$

and the range of f_1 is $\{\{a\},\{b\},\{c\}\}$, which does not contain $\emptyset$.
(ii) $f_2 : a \mapsto \emptyset$, $b \mapsto \{a\}$, $c \mapsto \{a,b\}$. Test each element against its own image:

$$\begin{aligned} a \notin \emptyset = f_2(a) \quad &\Longrightarrow \quad a \in B, \\ b \notin \{a\} = f_2(b) \quad &\Longrightarrow \quad b \in B, \\ c \notin \{a,b\} = f_2(c) \quad &\Longrightarrow \quad c \in B, \end{aligned}$$

so $B = \{a,b,c\} = A$, while the range of f_2 is $\{\emptyset,\{a\},\{a,b\}\}$ – no A there.
(iii) $g : C \to P(C)$, $g(k) = \{k\}$, on $C = \{1,2,3,4\}$. As in (i), $k \in g(k)$ for every k, so

$$B = \emptyset,$$

which is not among $\{1\},\{2\},\{3\},\{4\}$. ■

Problem 1.6.8. This exercise completes the proof of Theorem 1.6.2 (Cantor's Theorem): given any set A, there does not exist a function $f : A \to P(A)$ that is onto. In the proof one assumes for contradiction that $f : A \to P(A)$ is onto and forms the subset

$$B = \{a \in A : a \notin f(a)\}.$$

Since f is assumed onto and $B \subseteq A$, we must have $B = f(a')$ for some $a' \in A$. The contradiction arises when we consider whether or not a' is an element of B.
(a) First, show that the case $a' \in B$ leads to a contradiction.
(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution. Both cases fail, because substituting $f(a') = B$ into the rule defining B produces the self-contradiction

$$a' \in B \iff a' \notin f(a') \iff a' \notin B.$$

(a) If $a' \in B$, then by the definition of B we have $a' \notin f(a') = B$, contradicting $a' \in B$.
(b) If $a' \notin B = f(a')$, then a' meets the defining condition $a' \notin f(a')$, so $a' \in B$, again a contradiction.
Hence no $a' \in A$ satisfies $f(a') = B$, and since $B \in P(A)$ the arbitrary f is not onto.

Problem 1.6.9. Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution. Chain three equivalences: $P(\mathbf{N}) \sim S \sim (0,1) \sim \mathbf{R}$, where S is the set of sequences of 0's and 1's from Exercise 1.6.4. Since $\sim$ is transitive (Exercise 1.5.5 (c)), this suffices.
For $P(\mathbf{N}) \sim S$, send a subset to its characteristic sequence:

$$\chi(A) = (a_1, a_2, a_3, \ldots), \qquad a_n = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

The map $(a_n) \mapsto \{n \in \mathbf{N} : a_n = 1\}$ is a two-sided inverse, so χ is a 1–1, onto map from $P(\mathbf{N})$ to S.
For $S \sim (0,1)$, build a 1–1 map each way and apply the Schroder–Bernstein Theorem (Exercise 1.5.11), whose only hypothesis is that pair of injections.
Define $g : S \to (0,1)$ by

$$g\big((a_n)\big) = .b_1b_2b_3\ldots, \qquad b_n = \begin{cases} 1 & \text{if } a_n = 0, \\ 2 & \text{if } a_n = 1. \end{cases}$$

Every digit lies in $\{1,2\}$, so the value sits between $.111\ldots = 1/9$ and $.222\ldots = 2/9$ and hence in $(0,1)$. Decimal ambiguity is confined to terminating expansions versus repeating 9's (Exercise 1.6.3 (b)), and these digit strings are neither, so each is the only expansion of its value. Distinct sequences therefore give distinct reals, and g is 1–1.

Define $h : (0,1) \to S$ by taking the decimal expansion $x = .c_1c_2c_3\ldots$, choosing the non-terminating representation when there is a choice, and setting $h(x)$ to be the 0–1 sequence

$$\underbrace{1\cdots1}_{c_1+1}\,0\,\underbrace{1\cdots1}_{c_2+1}\,0\,\underbrace{1\cdots1}_{c_3+1}\,0\,\cdots.$$

Each block of 1's has length $c_n + 1 \geq 1$, so the blocks are nonempty and the zeros separating them are unambiguous: reading $h(x)$ left to right recovers $c_1, c_2, c_3, \ldots$ and hence x. So h is 1–1. Schroder–Bernstein now yields a 1–1, onto map between S and $(0,1)$.
Finally $(0,1) \sim \mathbf{R}$ is Exercise 1.5.4 (a) with $(a,b) = (0,1)$. Composing the three, $P(\mathbf{N}) \sim \mathbf{R}$.

Problem 1.6.10. As a final exercise, answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.
(a) Is the set of all functions from $\{0,1\}$ to $\mathbf{N}$ countable or uncountable?
(b) Is the set of all functions from $\mathbf{N}$ to $\{0,1\}$ countable or uncountable?
(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an antichain if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution. (a) Countable. The map $f \mapsto (f(0), f(1))$ is a 1–1 correspondence with $\mathbf{N} \times \mathbf{N}$, since a function on the two-point domain $\{0,1\}$ is exactly a choice of its two values. And

$$\mathbf{N} \times \mathbf{N} = \bigcup_{m=1}^{\infty} \left(\{m\} \times \mathbf{N}\right)$$

is countable by Theorem 1.5.8 (ii): the union is indexed by $m \in \mathbf{N}$ and each $\{m\} \times \mathbf{N} \sim \mathbf{N}$ via $(m,n) \mapsto n$, so both hypotheses hold.
(b) Uncountable. The map $f \mapsto \big(f(1), f(2), f(3), \ldots\big)$ is a 1–1 correspondence with the set S of sequences of 0's and 1's of Exercise 1.6.4, since a function into $\{0,1\}$ is nothing but the list of its values; and S is uncountable by Exercise 1.6.4.
(c) Yes. Let F be the set of all nonempty finite strings of 0's and 1's, and for a sequence $s = (s_1, s_2, s_3, \ldots) \in S$ let

$$A_s = \{\, s_1s_2\cdots s_k \in F : k \in \mathbf{N} \,\}$$

be the set of initial segments of s. Then $\mathcal{A}_0 = \{A_s : s \in S\}$ is an uncountable antichain in $P(F)$.
It is an antichain, and $s \mapsto A_s$ is 1–1, for the same one-line reason. If $s \neq t$, let k be the first index with $s_k \neq t_k$. The string $s_1s_2\cdots s_k$ lies in A_s, while the only length-k string in A_t is $t_1t_2\cdots t_k$, which differs from it in the kth place. Hence

$$A_s \not\subseteq A_t \quad \text{and, symmetrically,} \quad A_t \not\subseteq A_s.$$

So distinct sequences give incomparable (in particular distinct) sets, $s \mapsto A_s$ is a 1–1 correspondence of S onto $\mathcal{A}_0$, and $\mathcal{A}_0$ is uncountable because S is (Exercise 1.6.4).
Transfer this to $\mathbf{N}$ along the explicit bijection $\phi : F \to \mathbf{N}$,

$$\phi(w_1w_2\cdots w_k) = 2^k + \sum_{i=1}^{k} w_i 2^{k-i} - 1,$$

which reads $1w_1w_2\cdots w_k$ as a binary numeral and subtracts 1. It carries the 2^k strings of length k onto the block $\{2^k - 1, \ldots, 2^{k+1} - 2\}$, and these blocks partition $\mathbf{N}$, so ϕ is 1–1 and onto. (Check!) Put

$$\mathcal{A} = \big\{\phi(A_s) : s \in S\big\} \subseteq P(\mathbf{N}), \qquad \phi(A_s) = \{\phi(w) : w \in A_s\}.$$

Because ϕ is a bijection, $\phi(A_s) \subseteq \phi(A_t)$ holds precisely when $A_s \subseteq A_t$; so $\mathcal{A}$ is again an antichain, and $A_s \mapsto \phi(A_s)$ carries $\mathcal{A}_0$ onto $\mathcal{A}$ in a 1–1 fashion. Thus $\mathcal{A} \sim S$ is an uncountable antichain in $P(\mathbf{N})$.

2 Sequences and Series

Contents

2.1 Exercises 2.2.1–2.2.7

Problem 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?
Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.
Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence vercongе to two different values? What exactly is being described in this strange definition?

Solution. Vercongence to x is just boundedness: (x_n) verconges to x if and only if (x_n) is a bounded sequence, and in that case it verconges to every real number.
Taking $N = 1$ in the definition collapses it, since the clause must hold for *all* N:

$$(x_n) \text{ verconges to } x \iff \exists\, \epsilon > 0 \text{ with } |x_n - x| < \epsilon \text{ for all } n \in \mathbf{N}.$$

So the whole sequence lies in the single neighborhood $V_\epsilon(x)$ (Definition 2.2.4); no tail behaviour is constrained at all. If such an ϵ exists then $|x_n| < |x| + \epsilon$ for every n, so (x_n) is bounded; conversely if $|x_n| \leq M$ for all n, then for any $y \in \mathbf{R}$ the choice $\epsilon = M + |y| + 1$ gives $|x_n - y| \leq |x_n| + |y| < \epsilon$.
Examples, in order:

- $x_n = 1$ verconges to 1 (any $\epsilon > 0$ works).
- $x_n = (-1)^n$ verconges to 0 with $\epsilon = 2$, and is divergent: for any candidate limit a, one of $|1 - a|$, $|-1 - a|$ is at least 1, and that value recurs past every N, so $\epsilon = 1$ has no response.
- Yes: $(-1)^n$ verconges to 0 and to 17 (take $\epsilon = 19$), and indeed to every real number.

Problem 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.
(a) $\lim \dfrac{2n+1}{5n+4} = \dfrac{2}{5}$.
(b) $\lim \dfrac{2n^2}{n^3+3} = 0$.
(c) $\lim \dfrac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution. Take $N > 3/(25\epsilon)$ in (a), $N > 2/\epsilon$ in (b), $N > 1/\epsilon^3$ in (c); each proof then follows the template on p. 45.
(a) Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ with $N > 3/(25\epsilon)$. For $n \geq N$,

$$\begin{aligned}\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| \\ &= \frac{3}{25n+20} < \frac{3}{25n} \leq \frac{3}{25N} < \epsilon.\end{aligned}$$

(b) Let $\epsilon > 0$ and choose $N > 2/\epsilon$. For $n \geq N$, using $n^3 + 3 > n^3$,

$$\begin{aligned}\left|\frac{2n^2}{n^3+3} - 0\right| &= \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} \\ &= \frac{2}{n} \leq \frac{2}{N} < \epsilon.\end{aligned}$$

(c) Let $\epsilon > 0$ and choose $N > 1/\epsilon^3$. For $n \geq N$, since $|\sin(n^2)| \leq 1$,

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon,$$

the last step because $N > \epsilon^{-3}$ gives $\sqrt[3]{N} > 1/\epsilon$.

Problem 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.
(a) At every college in the United States, there is a student who is at least seven feet tall.
(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
(c) There exists a college in the United States where every student is at least six feet tall.

Solution. (a) Exhibit a single college in the United States at which *every* student is shorter than seven feet — by the rule on p. 46, the negation of "for all P, there exists Q" is "for at least one P, no Q is possible."
(b) Exhibit a single college and then, for *each* professor there, produce one student of that professor who received a grade other than A or B. The negation runs two levels deep:

$$\begin{aligned} &\neg(\forall\,\text{colleges}\ \exists\,\text{prof}\ \forall\,\text{students}) \\ &\qquad \equiv\ \exists\,\text{college}\ \forall\,\text{profs}\ \exists\,\text{student}. \end{aligned}$$

(c) Show that at *every* college in the United States there is some student shorter than six feet: the original claim is an existence claim, so a single college will not do, and one short student must be supplied for each college in the country.

Problem 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.
(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution. (a) Possible: $a_n = 1$ for n odd, $a_n = 0$ for n even, i.e. $(1, 0, 1, 0, 1, 0, \dots)$. There are infinitely many ones, but taking $\epsilon = 1/2$, no N works, since every N is followed by an even index $n \geq N$ with $|a_n - 1| = 1 > 1/2$.
(b) Impossible. Suppose $(a_n) \to a$ with $a \neq 1$ and $a_n = 1$ for infinitely many n. Apply Definition 2.2.3 with $\epsilon = |a-1|/2 > 0$ to get N with $|a_n - a| < |a-1|/2$ for all $n \geq N$. Some index $n \geq N$ has $a_n = 1$ (infinitely many do), and for that index

$$|a - 1| = |a_n - a| < \frac{|a-1|}{2},$$

forcing $|a - 1| < 0$, a contradiction.
(c) Possible: string together blocks of ones of increasing length, separated by single zeros,

$$(1,\ 0,\ 1, 1,\ 0,\ 1, 1, 1,\ 0,\ 1, 1, 1, 1,\ 0,\ \dots),$$

whose k-th block consists of k consecutive ones. Given $n \in \mathbf{N}$, the n-th block supplies n consecutive ones. The sequence diverges: both the value 1 and the value 0 occur at arbitrarily large indices, so for any candidate limit a the choice $\epsilon = 1/2$ admits no N — one of $|1 - a|$, $|0 - a|$ is at least $1/2$, and that value recurs past every N.

Problem 2.2.5. Let $[\![x]\!]$ be the greatest integer less than or equal to x. For example, $[\![\pi]\!] = 3$ and $[\![3]\!] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.
(a) $a_n = [\![5/n]\!]$,
(b) $a_n = [\![(12 + 4n)/3n]\!]$.
Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution. (a) $\lim a_n = 0$; (b) $\lim a_n = 1$. Both sequences are eventually constant, so a single N answers every ϵ at once.
(a) The first terms are $5, 2, 1, 1, 1, 0, 0, \ldots$, and for $n \geq 6$ we have $0 < 5/n \leq 5/6 < 1$, hence $a_n = 0$. Let $\epsilon > 0$ and take $N = 6$. For $n \geq N$,

$$|a_n - 0| = 0 < \epsilon.$$

(b) Write $a_n = [\![4/n + 4/3]\!]$. The first terms are $5, 3, 2, 2, 2, 2, 1, 1, \ldots$; for $n \geq 7$,

$$1 < \frac{4}{3} \leq \frac{4}{n} + \frac{4}{3} \leq \frac{4}{7} + \frac{4}{3} = \frac{40}{21} < 2,$$

so $a_n = 1$. Let $\epsilon > 0$ and take $N = 7$. For $n \geq N$, $|a_n - 1| = 0 < \epsilon$. (At $n = 6$ the value is exactly 2, so $N = 7$ is the smallest choice that works.)
The quoted remark says only that N *may* have to grow, and these sequences are the extreme case where it does not: $N = 6$ (resp. $N = 7$) answers $\epsilon = 1$ and $\epsilon = 10^{-100}$ alike. All that is guaranteed is that an N working for ϵ also works for every $\epsilon' > \epsilon$, so the least admissible N is nondecreasing in $1/\epsilon$ — not strictly increasing, and not necessarily unbounded.

Problem 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution. Two limits a and b satisfy $|a - b| < \epsilon$ for every $\epsilon > 0$, so $a = b$ by Theorem 1.2.6.
Assume $(a_n) \to a$ and $(a_n) \to b$, and fix $\epsilon > 0$. Definition 2.2.3 applied with $\epsilon/2 > 0$ supplies N_1 with $|a_n - a| < \epsilon/2$ for all $n \geq N_1$, and N_2 with $|a_n - b| < \epsilon/2$ for all $n \geq N_2$. Taking $n = \max\{N_1, N_2\}$, the triangle inequality (Example 1.2.5) gives

$$|a - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As $\epsilon > 0$ was arbitrary, Theorem 1.2.6 yields $a = b$.

Problem 2.2.7. Here are two useful definitions:
(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.
(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.
(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. (a) Frequently, but not eventually. Given any N, the even index $n = 2N \geq N$ has $(-1)^n = 1 \in \{1\}$; on the other hand no N works for "eventually," since every N is followed by an odd index n with $(-1)^n = -1 \notin \{1\}$.
(b) *Eventually* is stronger, and eventually implies frequently. If $a_n \in A$ for all $n \geq N_0$, then given any $N \in \mathbf{N}$ the index $n = \max\{N, N_0\}$ satisfies $n \geq N$ and $a_n \in A$. The converse fails by part (a).
(c) Definition 2.2.3B becomes: a sequence (a_n) converges to a if, for every $\epsilon > 0$, (a_n) is eventually in the ϵ-neighborhood $V_\epsilon(a)$.
"Eventually" is the term we want. Replacing it by "frequently" gives a strictly weaker and useless condition — by (a), $((-1)^n)$ is frequently in $V_\epsilon(1)$ for every $\epsilon > 0$, yet it does not converge to 1.
(d) Not necessarily eventually, but always frequently. For the first, $x_n = 2$ for n odd and $x_n = 0$

for n even has infinitely many terms equal to 2, while every N is followed by an even index n with $x_n = 0 \notin (1.9, 2.1)$. For the second, let $N \in \mathbf{N}$; the set $\{n : x_n = 2\}$ is infinite, hence not bounded above by N, so some $n \geq N$ has $x_n = 2 \in (1.9, 2.1)$.

2.2 Exercises 2.2.8–2.3.6

Problem 2.2.8. For some additional practice with nested quantifiers, consider the following invented definition:
Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.
(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero-heavy?
(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.
(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.
(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ....

Solution. (a) Yes, with $M = 1$. Here $x_n = 0$ for odd n and $x_n = 1$ for even n, and every pair of consecutive indices $\{N, N+1\}$ contains an odd one: take $n = N$ if N is odd, $n = N + 1$ if N is even. Either way $N \leq n \leq N + 1$ and $x_n = 0$.
(b) Yes. Let M witness zero-heaviness and set $N_k = k(M + 1) + 1$ for $k = 0, 1, 2, \ldots$. Applying the definition to each N_k produces an index n_k with

$$k(M+1)+1 \;\leq\; n_k \;\leq\; (k+1)(M+1), \qquad x_{n_k} = 0.$$

These windows are pairwise disjoint, so the n_k are distinct and $\{n : x_n = 0\}$ is infinite.
(c) No: the zeros can thin out faster than any fixed window. Take

$$x_n = \begin{cases} 0, & n = 2^k \text{ for some } k \in \mathbf{N}, \\ 1, & \text{otherwise,} \end{cases}$$

which has infinitely many zeros. Given any $M \in \mathbf{N}$, choose k with $2^k > M + 1$ and put $N = 2^k + 1$. The zeros nearest N sit at $2^k < N$ and at 2^{k+1}, and

$$2^{k+1} \;=\; (2^k + 1) + (2^k - 1) \;>\; N + M,$$

so no n with $N \leq n \leq N + M$ has $x_n = 0$. Hence no M works and (x_n) is not zero-heavy.
(d) Negating the quantifier string $(\exists M)(\forall N)(\exists n)$ term by term: a sequence (x_n) is not zero-heavy if for all $M \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that for all n satisfying $N \leq n \leq N + M$ we have $x_n \neq 0$.

Problem 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.
(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.
(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution. (a) Given $\epsilon > 0$, apply Definition 2.2.3 with the positive number ϵ^2: there is an N such that $n \geq N$ implies $x_n = |x_n - 0| < \epsilon^2$, and hence

$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \epsilon.$$

(b) First, $x \geq 0$ by the Order Limit Theorem 2.3.4(i) (the hypothesis $x_n \geq 0$ holds for all n), so $\sqrt{x}$ is defined.
(i) $x = 0$: this is part (a).

(ii) $x > 0$: rationalizing the numerator gives, for every n,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}},$$

the inequality because $\sqrt{x_n} \geq 0$. Given $\epsilon > 0$, choose N so that $n \geq N$ implies $|x_n - x| < \epsilon\sqrt{x}$; then $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ for all $n \geq N$.

Problem 2.3.2. Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then
(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;
(b) $(1/x_n) \to 1/2$.
(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution. (a) Given $\epsilon > 0$, choose N with $|x_n - 2| < \frac{3}{2}\epsilon$ for $n \geq N$. Then for such n,

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon.$$

(b) The quantity to control is

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|2 - x_n|}{2|x_n|},$$

so we need $|x_n|$ bounded away from 0. Applying Definition 2.2.3 with $\epsilon_0 = 1$ gives an N_1 such that $|x_n - 2| < 1$ for $n \geq N_1$; then $x_n > 1$, so $|x_n| > 1$.
Now let $\epsilon > 0$ and choose N_2 with $|x_n - 2| < 2\epsilon$ for $n \geq N_2$. Put $N = \max\{N_1, N_2\}$. For $n \geq N$,

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|2 - x_n|}{2|x_n|} < \frac{2\epsilon}{2} = \epsilon.$$

Problem 2.3.3. (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Given $\epsilon > 0$, Definition 2.2.3 supplies N_1 with $|x_n - l| < \epsilon$ for $n \geq N_1$, and N_2 with $|z_n - l| < \epsilon$ for $n \geq N_2$. Set $N = \max\{N_1, N_2\}$; for $n \geq N$ the hypothesis $x_n \leq y_n \leq z_n$ gives

$$l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon,$$

that is, $|y_n - l| < \epsilon$. Hence $\lim y_n = l$.

Problem 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):
(a) $\lim\left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2}\right)$
(b) $\lim\left(\frac{(a_n + 2)^2 - 4}{a_n}\right)$
(c) $\lim\left(\frac{\frac{2}{a_n} + 3}{\frac{1}{a_n} + 5}\right)$.

Solution. (a) 1. By Theorem 2.3.3(iii), $(a_n^2) \to 0$; then by (i) and (ii),

$$(1 + 2a_n) \to 1, \qquad (1 + 3a_n - 4a_n^2) \to 1.$$

Since the limit of the denominator is $1 \neq 0$, Theorem 2.3.3(iv) applies and the quotient converges to $1/1 = 1$.

(b) 4. The fractions are assumed defined, so $a_n \neq 0$ and we may cancel before taking limits:

$$\frac{(a_n+2)^2-4}{a_n} = \frac{a_n^2+4a_n}{a_n} = a_n + 4.$$

By Theorem 2.3.3(ii) this tends to $0+4=4$.
(c) 2. Again $a_n \neq 0$, so multiplying numerator and denominator by a_n is legitimate termwise:

$$\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} = \frac{2+3a_n}{1+5a_n}.$$

By Theorem 2.3.3(i),(ii) the numerator tends to 2 and the denominator to $1 \neq 0$, so by (iv) the limit is 2.

Problem 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution. Write the shuffle explicitly: $z_{2n-1} = x_n$ and $z_{2n} = y_n$ for all $n \in \mathbf{N}$.
($\Rightarrow$) Suppose $(z_n) \to l$ and let $\epsilon > 0$. Choose N so that $|z_m - l| < \epsilon$ for all $m \geq N$. If $n \geq N$ then both $2n-1 \geq n \geq N$ and $2n \geq n \geq N$, so

$$|x_n - l| = |z_{2n-1} - l| < \epsilon, \qquad |y_n - l| = |z_{2n} - l| < \epsilon.$$

Hence (x_n) and (y_n) both converge, and both to the same value l.
($\Leftarrow$) Suppose $\lim x_n = \lim y_n = l$, and let $\epsilon > 0$. Choose N_1 with $|x_n - l| < \epsilon$ for $n \geq N_1$ and N_2 with $|y_n - l| < \epsilon$ for $n \geq N_2$, and set

$$N = \max\{2N_1 - 1,\ 2N_2\}.$$

Let $m \geq N$. If $m = 2n-1$ is odd, then $2n-1 \geq 2N_1 - 1$ gives $n \geq N_1$, so $|z_m - l| = |x_n - l| < \epsilon$. If $m = 2n$ is even, then $2n \geq 2N_2$ gives $n \geq N_2$, so $|z_m - l| = |y_n - l| < \epsilon$. Thus $(z_n) \to l$.

Problem 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution. $\lim b_n = -1$. Rationalize the numerator and then divide through by n:

$$\begin{aligned} b_n &= \frac{n^2 - (n^2+2n)}{n + \sqrt{n^2+2n}} = \frac{-2n}{n+\sqrt{n^2+2n}} \\ &= \frac{-2}{1+\sqrt{1+2/n}}, \end{aligned}$$

the last step using $\sqrt{n^2+2n} = n\sqrt{1+2/n}$ (valid since $n > 0$).
Now $(1+2/n) \to 1$ by Theorem 2.3.3(i),(ii), and $1+2/n \geq 0$, so Exercise 2.3.1(b) gives $\left(\sqrt{1+2/n}\right) \to 1$. Hence the denominator $\left(1+\sqrt{1+2/n}\right) \to 2 \neq 0$, and Theorem 2.3.3(iv) yields

$$\lim b_n = \frac{-2}{2} = -1.$$

2.3 Exercises 2.3.7–2.3.13

Problem 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):
(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;
(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;
(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;
(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;
(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution. (a) Possible: $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Each diverges: consecutive terms differ by 2, so no single l can have all terms eventually within $\epsilon = 1$ of it. Meanwhile $x_n + y_n = 0$ for all n, so the sum converges to 0.
(b) Impossible. If $(x_n) \to x$ and $(x_n + y_n) \to s$, then by Theorem 2.3.3(i),(ii) applied to $y_n = (x_n + y_n) + (-1)x_n$,

$$\lim y_n = s - x,$$

so (y_n) converges, contradicting its divergence.
(c) Possible: $b_n = 1/n$. Then $b_n \neq 0$ and $(b_n) \to 0$, while $1/b_n = n$ is unbounded, hence divergent by Theorem 2.3.2. (No contradiction with Theorem 2.3.3(iv), whose hypothesis $b \neq 0$ fails here.)
(d) Impossible. A convergent (b_n) is bounded by Theorem 2.3.2, say $|b_n| \le M_1$; if also $|a_n - b_n| \le M_2$ for all n, then

$$|a_n| \le |a_n - b_n| + |b_n| \le M_2 + M_1$$

for all n, so (a_n) is bounded, contradicting the hypothesis.
(e) Possible: $a_n = 0$ for all n and $b_n = (-1)^n$. Then $(a_n) \to 0$ and $a_n b_n = 0 \to 0$, while (b_n) diverges as in part (a). (Theorem 2.3.3(iv) cannot be used to recover $b_n = (a_n b_n)/a_n$, since $\lim a_n = 0$.)

Problem 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.
(a) Show $p(x_n) \to p(x)$.
(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution. (a) Write $p(t) = c_k t^k + c_{k-1} t^{k-1} + \cdots + c_1 t + c_0$ and run the Algebraic Limit Theorem (Theorem 2.3.3) three times.
First, $x_n^j \to x^j$ for every $j \ge 0$, by induction on j: the case $j = 0$ is the constant sequence 1, and if $x_n^j \to x^j$ then Theorem 2.3.3 (iii) applied to the two convergent sequences (x_n^j) and (x_n) gives

$$x_n^{j+1} = x_n^j \, x_n \longrightarrow x^j x = x^{j+1}.$$

Next, Theorem 2.3.3 (i) gives $c_j x_n^j \to c_j x^j$ for each j, and finitely many applications of Theorem 2.3.3 (ii) add these $k+1$ convergent sequences:

$$p(x_n) = \sum_{j=0}^{k} c_j x_n^j \longrightarrow \sum_{j=0}^{k} c_j x^j = p(x).$$

(b) Take

$$f(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ 1 & \text{if } t = 0, \end{cases} \qquad x_n = \tfrac{1}{n}.$$

Then $(x_n) \to 0$ and $f(x_n) = 0$ for every n, so $f(x_n) \to 0$, while $f(0) = 1$.

Problem 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?
(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?
(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution. (a) Pick $M > 0$ with $|a_n| \leq M$ for all n (Definition 2.3.1). Given $\epsilon > 0$, convergence of (b_n) to 0 supplies N with $|b_n| < \epsilon/M$ for all $n \geq N$, and then for all $n \geq N$

$$|a_n b_n - 0| = |a_n|\,|b_n| \leq M|b_n| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

The Algebraic Limit Theorem is unavailable because its hypothesis is that *both* sequences converge; here (a_n) is only assumed bounded, so the symbol $\lim a_n$ in the conclusion $\lim(a_n b_n) = (\lim a_n)(\lim b_n)$ need not name anything.
(b) Only this: with $b \neq 0$, the sequence $(a_n b_n)$ converges if and only if (a_n) does. One direction is Theorem 2.3.3 (iii); for the other, note $b \neq 0$ forces $|b_n| > |b|/2 > 0$ for all $n \geq N_0$ (take $\epsilon = |b|/2$ in the definition of $\lim b_n = b$), so the quotient below is defined from N_0 on, and Theorem 2.3.3 (iv) applies since $b \neq 0$:

$$a_n = \frac{a_n b_n}{b_n} \longrightarrow \frac{\lim(a_n b_n)}{b}.$$

In particular convergence of $(a_n b_n)$ is not automatic: $a_n = (-1)^n$, $b_n = 1$ gives $b = 1 \neq 0$ and $(a_n b_n) = ((-1)^n)$ divergent.
(c) This is the Algebraic Limit Theorem's product rule, $\lim(a_n b_n) = ab$, in the case $a = 0$. Since (b_n) converges it is bounded (Theorem 2.3.2), and $(a_n) \to a = 0$, so part (a) with the roles of the two sequences interchanged gives

$$\lim(a_n b_n) = 0 = 0 \cdot b = ab.$$

Problem 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.
(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
(b) If $(b_n) \to b$, then $|b_n| \to |b|$.
(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution. (a) False. Take $a_n = b_n = n$. Then $a_n - b_n = 0 \to 0$, but neither $\lim a_n$ nor $\lim b_n$ exists, so the asserted equation has no meaning.
(b) True, and no ϵ is needed beyond the reverse triangle inequality of Exercise 1.2.6 (d):

$$\big||b_n| - |b|\big| \leq |b_n - b|.$$

Given $\epsilon > 0$, any N that works for $(b_n) \to b$ works here.
(c) True. Both (a_n) and $(b_n - a_n)$ converge, so Theorem 2.3.3 (ii) applies to their sum:

$$b_n = a_n + (b_n - a_n) \longrightarrow a + 0 = a.$$

(d) True. The hypothesis forces $a_n \geq |b_n - b| \geq 0$. Given $\epsilon > 0$, choose N with $|a_n - 0| < \epsilon$ for all $n \geq N$; then for those n

$$|b_n - b| \leq a_n = |a_n| < \epsilon.$$

Problem 2.3.11. (Cesaro Means).

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.
(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution. (a) Say $(x_n) \to x$; split the average at the index past which the tail is already $\epsilon/2$-close. Since (x_n) converges it is bounded (Theorem 2.3.2), say $|x_k| \le B$ for all k, so the set $\{\, |x_k - x| : k \in \mathbf{N} \,\}$ is nonempty and bounded above by $B + |x|$; by the Axiom of Completeness it has a supremum

$$M = \sup\{\, |x_k - x| : k \in \mathbf{N} \,\}.$$

Let $\epsilon > 0$ and choose N_1 with $|x_k - x| < \epsilon/2$ for all $k \ge N_1$. For $n \ge N_1$, writing $y_n - x = \frac{1}{n}\sum_{k=1}^{n}(x_k - x)$ and applying the triangle inequality,

$$\begin{aligned} |y_n - x| &\le \frac{1}{n}\sum_{k=1}^{N_1-1} |x_k - x| + \frac{1}{n}\sum_{k=N_1}^{n} |x_k - x| \\ &\le \frac{(N_1 - 1)M}{n} + \frac{1}{n}\cdot n \cdot \frac{\epsilon}{2} \\ &= \frac{(N_1 - 1)M}{n} + \frac{\epsilon}{2}. \end{aligned}$$

The numerator $(N_1 - 1)M$ is a constant, so by the Archimedean Property (Theorem 1.4.2) there is $N_2 \in \mathbf{N}$ with $(N_1 - 1)M/N_2 < \epsilon/2$. Setting $N = \max\{N_1, N_2\}$ gives $|y_n - x| < \epsilon$ for all $n \ge N$.
(b) Take $x_n = (-1)^n$, which diverges. The partial sums alternate between -1 and 0, so

$$y_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{1}{n} & \text{if } n \text{ is odd,} \end{cases}$$

whence $|y_n| \le 1/n$ and $(y_n) \to 0$ by Exercise 2.3.10 (d).

Problem 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.
(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.
(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.
(c) If every a_n is rational, then a is rational.

Solution. (a) True. Fix $b \in B$. Then $b \le a_n$ for every $n \in \mathbf{N}$, so the Order Limit Theorem (Theorem 2.3.4 (iii), with the constant $c = b$) gives $b \le a$. As $b \in B$ was arbitrary, a is an upper bound for B.
(b) True. Suppose instead $a \in (0, 1)$ and set

$$\epsilon = \min\{a,\, 1 - a\} > 0.$$

Convergence supplies N with $|a_N - a| < \epsilon$, and then

$$a_N > a - \epsilon \ge 0, \qquad a_N < a + \epsilon \le 1,$$

so $a_N \in (0, 1)$, contradicting the hypothesis.
(c) False. By the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) choose a rational a_n with

$$\sqrt{2} - \frac{1}{n} < a_n < \sqrt{2},$$

where $\sqrt{2}$ denotes the real number with square 2 supplied by Theorem 1.4.5, and $\sqrt{2} \notin \mathbf{Q}$ by Theorem 1.1.1. Then $|a_n - \sqrt{2}| < 1/n$, so $(a_n) \to \sqrt{2}$ by Exercise 2.3.10 (d) while every a_n is rational.

Problem 2.3.13. (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?
(a) Let $a_{mn} = m/(m+n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.
(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.
(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.
(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.
(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution. It should represent a single number a that every entry a_{mn} approximates once *both* indices are large – precisely the ϵ-N definition quoted in part (a), in which one N must serve for all pairs (m, n) with $m, n \geq N$.
(a) The two iterated limits are 1 and 0. For fixed n, dividing by m gives $a_{mn} = 1/(1 + n/m) \to 1$ as $m \to \infty$ (Theorem 2.3.3 (iv), the denominator tending to $1 \neq 0$), so the first iterated limit is $\lim_{n\to\infty} 1 = 1$. For fixed m, $a_{mn} = m/(m+n)$ satisfies $0 < a_{mn} \leq m/n \to 0$, so $\lim_{n\to\infty} a_{mn} = 0$ and the second iterated limit is 0.
(b) For $a_{mn} = 1/(m+n)$ all three limits exist and equal 0. Given $\epsilon > 0$ pick $N > 1/(2\epsilon)$ (Theorem 1.4.2); if $m, n \geq N$ then

$$|a_{mn} - 0| = \frac{1}{m+n} \leq \frac{1}{2N} < \epsilon,$$

so $\lim_{m,n\to\infty} a_{mn} = 0$; and each inner limit is 0 by the same bound with one index fixed, so both iterated limits are $\lim 0 = 0$.
For $a_{mn} = mn/(m^2+n^2)$ the two iterated limits exist and equal 0, but the double limit does not exist. For fixed m, $0 < a_{mn} \leq mn/n^2 = m/n \to 0$, so $\lim_{n\to\infty} a_{mn} = 0$ and hence $\lim_m \lim_n a_{mn} = 0$; by symmetry of a_{mn} in m and n the other iterated limit is also 0. If the double limit were some a, take $\epsilon = 1/40$ and the corresponding N; the pairs (N, N) and $(2N, N)$ both qualify, and

$$a_{NN} = \tfrac{1}{2}, \qquad a_{2N,N} = \frac{2N^2}{5N^2} = \tfrac{2}{5},$$

so $\frac{1}{10} = |a_{NN} - a_{2N,N}| \leq |a_{NN} - a| + |a - a_{2N,N}| < \frac{1}{20}$, a contradiction.
(c) Take

$$a_{mn} = \frac{(-1)^m}{n} + \frac{(-1)^n}{m}.$$

The double limit is 0: if $m, n \geq N$ then $|a_{mn}| \leq 1/n + 1/m \leq 2/N$, which is $< \epsilon$ once $N > 2/\epsilon$. But for each fixed m the inner limit in n fails to exist: $(-1)^m/n \to 0$ as $n \to \infty$, so if $(a_{mn})_n$ converged, then by Theorem 2.3.3 (i) and (ii) so would

$$a_{mn} - \frac{(-1)^m}{n} = \frac{(-1)^n}{m},$$

whose terms take the two values $\pm 1/m$ for arbitrarily large n; no single L lies within $1/m$ of both, so this sequence has no limit. Since a_{mn} is symmetric in m and n, the inner limit in m fails as well, and neither iterated limit can be formed.

(d) Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ with $|a_{mn} - a| < \epsilon/2$ whenever $m, n \geq N$. Fix $m \geq N$. Since $(a_{mn})_n \to b_m$, Theorem 2.3.3 (ii) gives $a_{mn} - a \to b_m - a$, and then Exercise 2.3.10 (b) gives

$$|a_{mn} - a| \longrightarrow |b_m - a| \qquad (n \to \infty).$$

Every term of this convergent sequence with index $n \geq N$ satisfies $|a_{mn} - a| \leq \epsilon/2$, and discarding the finitely many terms with $n < N$ changes neither the sequence's limit nor the hypothesis of the Order Limit Theorem (Exercise 2.2.7), so Theorem 2.3.4 (iii) yields

$$|b_m - a| \leq \frac{\epsilon}{2} < \epsilon.$$

As $m \geq N$ was arbitrary, $\lim_{m\to\infty} b_m = a$.
(e) Write $a = \lim_{m,n\to\infty} a_{mn}$. Saying that the iterated limit $\lim_m \lim_n a_{mn}$ exists includes the existence of $b_m = \lim_{n\to\infty} a_{mn}$ for each fixed m, so the hypotheses of (d) hold verbatim and

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m\to\infty} b_m = a.$$

For the other order, existence of the second iterated limit supplies $c_k = \lim_{m\to\infty} a_{mk}$ for each fixed k; apply (d) to the transposed array $\tilde{a}_{mn} = a_{nm}$. Its double limit is again a, since the defining condition $|a_{mn} - a| < \epsilon$ for all $m, n \geq N$ is symmetric in the two indices, and for each fixed m its inner limit is $\lim_{n\to\infty} \tilde{a}_{mn} = \lim_{n\to\infty} a_{nm} = c_m$. Hence (d) gives

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{n\to\infty} c_n = a.$$

2.4 Exercises 2.4.1–2.4.7

Problem 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.
(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.
(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution. $\lim x_n = 2 - \sqrt{3}$.
(a) The interval $I = [2 - \sqrt{3}, 3]$ is invariant. Indeed $t \mapsto 1/(4 - t)$ is increasing on I (the denominator stays in $[1, 2 + \sqrt{3}]$, in particular never 0), so $x_n \in I$ gives

$$x_{n+1} \geq \frac{1}{4 - (2 - \sqrt{3})} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3},$$
$$x_{n+1} \leq \frac{1}{4 - 3} = 1 \leq 3.$$

Since $x_1 = 3 \in I$, induction gives $x_n \in I$ for all n. Monotonicity now follows from the sign of

$$x_{n+1} - x_n = \frac{1 - x_n(4 - x_n)}{4 - x_n} = \frac{(x_n - (2 - \sqrt{3}))(x_n - (2 + \sqrt{3}))}{4 - x_n},$$

whose denominator is positive and whose numerator has a nonnegative first factor and a negative second factor on I (as $2 + \sqrt{3} > 3$). So (x_n) is decreasing and bounded below by $2 - \sqrt{3}$, and the Monotone Convergence Theorem (Theorem 2.4.2) gives convergence.
(b) Write $L = \lim x_n$ and let $\varepsilon > 0$. Choose N with $|x_n - L| < \varepsilon$ for all $n \geq N$; if $n \geq N$ then also $n + 1 \geq N$, so $|x_{n+1} - L| < \varepsilon$. Hence $\lim x_{n+1} = L$. (Equivalently, (x_{n+1}) is a subsequence of (x_n), so Theorem 2.5.2 applies.)

(c) Taking limits across $x_{n+1}(4 - x_n) = 1$ and using the Algebraic Limit Theorem (Theorem 2.3.3) together with (b),

$$L(4 - L) = 1, \qquad L^2 - 4L + 1 = 0, \qquad L = 2 \pm \sqrt{3}.$$

Since $x_n \le 1$ for every $n \ge 2$, the Order Limit Theorem (Theorem 2.3.4) forces $L \le 1$, ruling out $2 + \sqrt{3}$. Thus $\lim x_n = 2 - \sqrt{3}$.

Problem 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.
What is wrong with this argument?
(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution. (a) The argument assumes what it must prove: that $\lim y_n$ exists. Here it does not. The recursion produces

$$1,\ 2,\ 1,\ 2,\ 1,\ 2,\ \ldots$$

so $y_n = 1$ for odd n and $y_n = 2$ for even n, and no ε-neighbourhood of radius $1/2$ about any candidate limit contains a tail. The algebra $y = 3 - y$ is a valid consequence of the recursion only once convergence is known; the number $3/2$ it produces is merely the fixed point of $t \mapsto 3 - t$, which this sequence never approaches.
(b) Yes, and now the strategy is legitimate, because convergence can be established first. Put $L = (3 + \sqrt{5})/2$, the positive root of $t^2 - 3t + 1 = 0$, so that $3 - 1/L = L$. The map $t \mapsto 3 - 1/t$ is increasing on $[1, L]$, and $y_1 = 1$, so induction gives $y_n \in [1, L]$ for all n:

$$1 \le y_n \le L \implies 2 = 3 - \tfrac{1}{1} \le y_{n+1} \le 3 - \tfrac{1}{L} = L.$$

On this range $y_n^2 - 3y_n + 1 \le 0$, since its roots are $(3 \pm \sqrt{5})/2$ and $(3 - \sqrt{5})/2 < 1 \le y_n \le L$. Hence

$$y_{n+1} - y_n = \frac{3y_n - 1 - y_n^2}{y_n} = \frac{-(y_n^2 - 3y_n + 1)}{y_n} \ge 0.$$

Thus (y_n) is increasing and bounded above, and the Monotone Convergence Theorem (Theorem 2.4.2) gives $y = \lim y_n$. Now the strategy applies: $y \ge 1 > 0$, so by the Algebraic Limit Theorem (Theorem 2.3.3),

$$y = 3 - \frac{1}{y}, \qquad y^2 - 3y + 1 = 0, \qquad y = \frac{3 \pm \sqrt{5}}{2}.$$

The root $(3 - \sqrt{5})/2 < 1$ is excluded by $y \ge 1$ (Theorem 2.3.4), so $\lim y_n = \frac{3+\sqrt{5}}{2}$.

Problem 2.4.3. (a) Show that

$$\sqrt{2},\ \sqrt{2 + \sqrt{2}},\ \sqrt{2 + \sqrt{2 + \sqrt{2}}},\ \ldots$$

converges and find the limit.
(b) Does the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

converge? If so, find the limit.

Solution. Both converge to 2.
(a) The sequence is $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$; every term is positive. It is bounded above by 2 by induction: $x_1 = \sqrt{2} < 2$, and $x_n < 2$ gives $x_{n+1} = \sqrt{2 + x_n} < \sqrt{4} = 2$. It is increasing because for $0 < x_n < 2$,

$$x_{n+1}^2 - x_n^2 = 2 + x_n - x_n^2 = -(x_n - 2)(x_n + 1) > 0,$$

and both terms are positive, so $x_{n+1} > x_n$. By the Monotone Convergence Theorem (Theorem 2.4.2), $L = \lim x_n$ exists, and $L \geq \sqrt{2} > 0$. Squaring the recursion and taking limits (Theorem 2.3.3, with $\lim x_{n+1} = L$ as in Exercise 2.4.1(b)),

$$L^2 = 2 + L, \qquad (L - 2)(L + 1) = 0, \qquad L = 2.$$

(b) Here $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$, and the argument of (a) repeats verbatim. Induction gives $0 < x_n < 2$ (from $\sqrt{2 \cdot 2} = 2$), and then

$$x_{n+1}^2 - x_n^2 = 2x_n - x_n^2 = x_n(2 - x_n) > 0,$$

so $x_{n+1} > x_n$, both terms being positive. Theorem 2.4.2 applies, and $L = \lim x_n$ satisfies $L^2 = 2L$ with $L \geq \sqrt{2} > 0$, whence $L = 2$.
Method (2): the recursion has the closed form

$$x_n = 2^{\,1-2^{-n}} \quad \text{since} \quad \left(2 \cdot 2^{\,1-2^{-n}}\right)^{1/2} = 2^{\,1-2^{-(n+1)}},$$

so $x_n \to 2^1 = 2$ by continuity of $t \mapsto 2^t$ (Chapter 4) and $2^{-n} \to 0$.

Problem 2.4.4. (a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.
(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.
These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution. Throughout, the Monotone Convergence Theorem (Theorem 2.4.2) is taken as the axiom; the Algebraic and Order Limit Theorems (Theorems 2.3.3, 2.3.4) are proved straight from the definition of convergence and use no AoC.
(a) Suppose, for contradiction, that $\mathbf{N}$ is bounded above in $\mathbf{R}$. The sequence $a_n = n$ is then increasing and bounded, so by Theorem 2.4.2 it converges, say $a_n \to a$; and $a_{n+1} \to a$ as well (Exercise 2.4.1(b)). By the Algebraic Limit Theorem,

$$1 = \lim(a_{n+1} - a_n) = a - a = 0,$$

a contradiction. Hence $\mathbf{N}$ is unbounded above, which is statement (i) of Theorem 1.4.2: for any $x \in \mathbf{R}$ there is $n \in \mathbf{N}$ with $n > x$. For statement (ii), given $y > 0$ pick $n > 1/y$; then $1/n < y$.
(b) Let $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$ for every n. Nesting says precisely

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1,$$

so (a_n) is increasing and bounded above by b_1. By Theorem 2.4.2 it converges; set $a = \lim a_n$. Fix n. For every $m \geq n$ we have $I_m \subseteq I_n$, hence

$$a_n \leq a_m \leq b_n.$$

Applying the Order Limit Theorem to the tail $(a_m)_{m \geq n}$, which has the same limit a, gives $a_n \leq a \leq b_n$, i.e. $a \in I_n$. As n was arbitrary,

$$a \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset. \qquad \blacksquare$$

Problem 2.4.5. (Calculating Square Roots). Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.
(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution. (a) All terms are positive ($x_1 = 2 > 0$, and $x_n > 0$ makes x_{n+1} an average of positive numbers), so the recursion is well posed. The key identity is

$$x_{n+1}^2 - 2 = \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) - 2 = \frac{1}{4}\left(x_n - \frac{2}{x_n}\right)^2 \;\geq\; 0,$$

so $x_n^2 \geq 2$ for every $n \geq 2$, and $x_1^2 = 4 \geq 2$ as well. Consequently

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{x_n^2 - 2}{2x_n} \;\geq\; 0.$$

Thus (x_n) is decreasing and bounded below by $\sqrt{2}$, and Theorem 2.4.2 gives $L = \lim x_n$ with $L \geq \sqrt{2} > 0$ (Theorem 2.3.4). Multiplying the recursion by $2x_n$ and passing to the limit (Theorem 2.3.3, with $\lim x_{n+1} = L$),

$$2L^2 = L^2 + 2, \qquad L^2 = 2, \qquad L = \sqrt{2}.$$

(b) For $c > 0$ take $x_1 = c$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

The same two displays hold verbatim with 2 replaced by c:

$$x_{n+1}^2 - c = \frac{1}{4}\left(x_n - \frac{c}{x_n}\right)^2 \geq 0, \qquad x_n - x_{n+1} = \frac{x_n^2 - c}{2x_n} \geq 0 \quad (n \geq 2),$$

so from the second term on, (x_n) decreases and is bounded below by $\sqrt{c}$. Theorem 2.4.2 and $2L^2 = L^2 + c$ give $\lim x_n = \sqrt{c}$.

Problem 2.4.6. (Arithmetic-Geometric Mean).
(a) Explain why $\sqrt{xy} \leq (x + y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)
(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \qquad \text{and} \qquad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution. (a) Square roots are real and the square of a real number is nonnegative:

$$0 \leq \left(\sqrt{x} - \sqrt{y}\right)^2 = x + y - 2\sqrt{xy} \quad\Longleftrightarrow\quad \sqrt{xy} \leq \frac{x+y}{2},$$

valid for all $x, y \geq 0$, with equality exactly when $x = y$.
(b) Part (a) applied to the pair (x_n, y_n) says $x_{n+1} \leq y_{n+1}$, and $x_1 \leq y_1$ is given, so

$$0 \leq x_n \leq y_n \qquad \text{for every } n \in \mathbf{N}.$$

(Nonnegativity propagates since x_{n+1} is a square root and y_{n+1} an average of nonnegative numbers.)

These inequalities make each sequence monotone:

$$x_{n+1} = \sqrt{x_n y_n} \;\ge\; \sqrt{x_n\, x_n} \;=\; x_n,$$
$$y_{n+1} = \frac{x_n + y_n}{2} \;\le\; \frac{y_n + y_n}{2} \;=\; y_n.$$

So (x_n) is increasing and bounded above by $x_n \le y_n \le y_1$, while (y_n) is decreasing and bounded below by $y_n \ge x_n \ge x_1$. The Monotone Convergence Theorem (Theorem 2.4.2) gives $x = \lim x_n$ and $y = \lim y_n$.
They agree: $\lim y_{n+1} = y$ (Exercise 2.4.1(b)), so taking limits in $2y_{n+1} = x_n + y_n$ via the Algebraic Limit Theorem (Theorem 2.3.3) yields

$$2y = x + y, \qquad \text{hence} \qquad x = y. \qquad \blacksquare$$

Problem 2.4.7. (Limit Superior). Let (a_n) be a bounded sequence.
(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \ge n\}$ converges.
(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.
(c) Prove that $\liminf a_n \le \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.
(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution. Fix $M > 0$ with $|a_n| \le M$ for all n, and write $A_n = \{a_k : k \ge n\}$.
(a) Each A_n is nonempty and bounded above by M, so $y_n = \sup A_n$ exists by the Axiom of Completeness. Since $A_{n+1} \subseteq A_n$, any upper bound for A_n is one for A_{n+1}, whence

$$y_{n+1} \le y_n \qquad \text{for all } n,$$

and $y_n \ge a_n \ge -M$. So (y_n) is decreasing and bounded below, and the Monotone Convergence Theorem (Theorem 2.4.2) gives convergence.
(b) Define

$$\liminf a_n = \lim z_n, \qquad z_n = \inf\{a_k : k \ge n\}.$$

Each z_n exists (the set A_n is bounded below by $-M$), and shrinking A_n to A_{n+1} can only raise the infimum, so (z_n) is increasing; it is bounded above by M. Theorem 2.4.2 again gives existence.
(c) For each n the set A_n is nonempty, so $z_n \le a_n \le y_n$; in particular $z_n \le y_n$, and the Order Limit Theorem (Theorem 2.3.4) gives

$$\liminf a_n = \lim z_n \;\le\; \lim y_n = \limsup a_n.$$

For strictness take $a_n = (-1)^n$: here $A_n = \{-1, 1\}$ for every n, so $z_n = -1$ and $y_n = 1$, giving $\liminf a_n = -1 < 1 = \limsup a_n$.
(d) ($\Leftarrow$) Suppose $a_n \to a$ and let $\varepsilon > 0$. Choose N so that $a - \varepsilon < a_k < a + \varepsilon$ for all $k \ge N$. For $n \ge N$ the number $a + \varepsilon$ is an upper bound for A_n and $a - \varepsilon$ a lower bound, so

$$a - \varepsilon < a_n \le y_n \le a + \varepsilon,$$
$$a + \varepsilon > a_n \ge z_n \ge a - \varepsilon.$$

Hence $|y_n - a| \le \varepsilon$ and $|z_n - a| \le \varepsilon$ for all $n \ge N$, so $\lim y_n = \lim z_n = a$; the two agree and share the value of $\lim a_n$.
($\Rightarrow$) Suppose $\lim z_n = \lim y_n = L$ and let $\varepsilon > 0$. Choose N_1, N_2 with $|y_n - L| < \varepsilon$ for $n \ge N_1$ and $|z_n - L| < \varepsilon$ for $n \ge N_2$. For $n \ge \max\{N_1, N_2\}$, the sandwich $z_n \le a_n \le y_n$ gives

$$L - \varepsilon < z_n \le a_n \le y_n < L + \varepsilon,$$

so $|a_n - L| < \varepsilon$. Thus $\lim a_n = L$, and all three quantities equal L. ■

2.5 Exercises 2.4.8–2.5.4

Problem 2.4.8. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (c) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$
(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution. (a) $s_m = 1 - 1/2^m$, so the series converges to 1. Indeed $2s_m - s_m$ telescopes:

$$\begin{aligned} 2s_m &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{m-1}}, \\ s_m &= \tfrac{1}{2} + \cdots + \tfrac{1}{2^{m-1}} + \tfrac{1}{2^m}, \end{aligned}$$

and subtracting gives $s_m = 1 - 2^{-m}$. Since $0 < 2^{-m} \le 1/m$, the Archimedean Property (Theorem 1.4.2) forces $2^{-m} \to 0$, so $s_m \to 1$.
(b) $s_m = 1 - \dfrac{1}{m+1}$, so the series converges to 1. Partial fractions give $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, and the sum telescopes:

$$s_m = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{m+1}.$$

(c) $s_m = \log(m+1)$, so the series diverges. Since log carries products to sums,

$$s_m = \sum_{n=1}^{m} \bigl(\log(n+1) - \log n\bigr) = \log(m+1),$$

again a telescoping sum. The sequence $(\log(m+1))$ is unbounded (given $M > 0$, take $m + 1 > e^M$), and convergent sequences are bounded by Theorem 2.3.2, so (s_m) diverges.

Problem 2.4.9. Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution. The partial sums satisfy $s_{2^k} \ge \frac{1}{2} t_k$, which is the whole content. Here (b_n) is decreasing with $b_n \ge 0$ (the hypotheses of Theorem 2.4.6), and

$$\begin{aligned} s_m &= b_1 + b_2 + \cdots + b_m, \\ t_k &= \sum_{n=0}^{k} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}. \end{aligned}$$

Group the first 2^k terms of (s_m) exactly as in Example 2.4.5, and bound each block $b_{2^{j-1}+1} + \cdots + b_{2^j}$ (which has 2^{j-1} terms, each $\ge b_{2^j}$ because (b_n) is decreasing) below by $2^{j-1} b_{2^j}$:

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\ &\ge b_1 + b_2 + (b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\ &\ge \tfrac{1}{2}\bigl(b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}\bigr) = \tfrac{1}{2}\, t_k. \end{aligned}$$

(Only the leading term is weakened: $b_1 \ge \frac{1}{2} b_1$, while every later term matches.)
Now suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. The partial sums (t_k) increase (each $2^n b_{2^n} \ge 0$), so the Monotone Convergence Theorem (Theorem 2.4.2) forbids them from being bounded. Hence $s_{2^k} \ge \frac{1}{2} t_k$ makes the subsequence (s_{2^k}), and with it (s_m), unbounded; convergent sequences are bounded (Theorem 2.3.2), so $\sum_{n=1}^{\infty} b_n$ diverges.

Problem 2.4.10. (Infinite Products). A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots,$$

where $a_n \geq 0$.
(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.
(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution. (a) $p_m = m + 1$, which diverges. The product telescopes:

$$p_m = \prod_{n=1}^{m}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{m}\frac{n+1}{n} = \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{m+1}{m} = m+1,$$

unbounded, hence divergent by Theorem 2.3.2.
For $a_n = 1/n^2$ the partial products $p_m = \prod_{n=1}^{m}\frac{n^2+1}{n^2}$ are

$$2,\quad \frac{5}{2},\quad \frac{25}{9},\quad \frac{425}{144},\quad \frac{221}{72},\quad \frac{8177}{2592},$$

that is, approximately 2, 2.500, 2.778, 2.951, 3.069, 3.155. The increments shrink rapidly; conjecture: this sequence converges.
(b) The sequence (p_m) is increasing, since $p_{m+1} = p_m(1+a_{m+1}) \geq p_m$ and $p_m \geq 1 > 0$. So by the Monotone Convergence Theorem (Theorem 2.4.2), (p_m) converges if and only if it is bounded, and likewise the increasing partial sums $s_m = \sum_{n=1}^{m} a_n$ converge if and only if they are bounded. It therefore suffices to show (p_m) is bounded if and only if (s_m) is.
($\Leftarrow$) Using $1 + x \leq 3^x$ for $x \geq 0$ termwise,

$$p_m = \prod_{n=1}^{m}(1+a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{s_m} \leq 3^{s},$$

where $s = \lim s_m = \sup_m s_m$ is finite when $\sum a_n$ converges (the last step uses that $x \mapsto 3^x$ is increasing). So (p_m) is bounded.
($\Rightarrow$) In the other direction,

$$p_m \geq 1 + s_m \qquad \text{for every } m,$$

by induction: equality holds at $m = 1$, and if $p_m \geq 1 + s_m$ then

$$\begin{aligned} p_{m+1} = p_m(1+a_{m+1}) &\geq (1+s_m)(1+a_{m+1}) \\ &= 1 + s_m + a_{m+1} + s_m a_{m+1} \\ &\geq 1 + s_{m+1}, \end{aligned}$$

since $s_m a_{m+1} \geq 0$. If (p_m) converges it is bounded by some M (Theorem 2.3.2), whence $s_m \leq M - 1$ for all m and (s_m) is bounded.

Problem 2.5.1. Give an example of each of the following, or argue that such a request is impossible.
(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.
(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

Solution. (a) Impossible. A bounded subsequence (x_{n_k}) is itself a bounded sequence, so the Bolzano–Weierstrass Theorem (Theorem 2.5.5) produces a convergent subsequence $(x_{n_{k_j}})$ of it; since $j \mapsto n_{k_j}$ is strictly increasing, this is a subsequence of (x_n) as well.
(b) Take

$$x_n = \begin{cases} \dfrac{1}{n+1}, & n \text{ odd}, \\ \dfrac{n}{n+1}, & n \text{ even}. \end{cases}$$

Every term lies in $(0, 1)$, so no term equals 0 or 1; the odd-indexed subsequence $(1/2, 1/4, 1/6, \ldots) \to 0$ and the even-indexed subsequence $(2/3, 4/5, 6/7, \ldots) \to 1$.
(c) List the finite blocks $1;\ 1, \frac{1}{2};\ 1, \frac{1}{2}, \frac{1}{3};\ \ldots$ one after another:

$$(x_n) = \Big(1,\ 1, \tfrac{1}{2},\ 1, \tfrac{1}{2}, \tfrac{1}{3},\ 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4},\ \ldots\Big).$$

For each m the value $1/m$ occurs in every block of length $\geq m$, hence at infinitely many indices; the constant subsequence built from those indices converges to $1/m$.
(d) Impossible. Suppose for each $m \in \mathbf{N}$ some subsequence of (x_n) converges to $1/m$; then for each m the inequality $|x_n - 1/m| < 1/m$ holds for infinitely many n. Choose indices recursively: having picked n_{m-1}, pick $n_m > n_{m-1}$ with $|x_{n_m} - 1/m| < 1/m$. Then

$$|x_{n_m} - 0| \leq |x_{n_m} - \tfrac{1}{m}| + \tfrac{1}{m} < \tfrac{2}{m},$$

so $(x_{n_m}) \to 0$. This is a subsequence of (x_n) converging to 0, which is not a point of $\{1, 1/2, 1/3, \ldots\}$.

Problem 2.5.2. Decide whether the following propositions are true or false, providing a short justification for each conclusion.
(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.
(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.
(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.
(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution. All four are true.
(a) The tail $(x_2, x_3, x_4, \ldots)$ is a proper subsequence, so it converges to some L: given $\epsilon > 0$ there is K with $|x_{k+1} - L| < \epsilon$ for all $k \geq K$. Then $n \geq K + 1$ gives $|x_n - L| < \epsilon$, so $(x_n) \to L$ by Definition 2.2.3.
(b) Contrapositive of Theorem 2.5.2: if (x_n) converged, every subsequence would converge (to the same limit).
(c) By Bolzano–Weierstrass (Theorem 2.5.5) the bounded sequence (x_n) has a subsequence $(x_{n_k}) \to L$. Since (x_n) does not converge to L, there is an $\epsilon_0 > 0$ with $|x_n - L| \geq \epsilon_0$ for infinitely many n; those indices give a bounded subsequence (x_{m_j}), and Theorem 2.5.5 again yields a further subsequence converging to some L'. Along it $|x_{m_j} - L| \to |L' - L|$ (Exercise 2.3.10(b) with Theorem 2.3.3), so the Order Limit Theorem (Theorem 2.3.4) gives $|L' - L| \geq \epsilon_0 > 0$ and $L \neq L'$.
(d) Say (x_n) is increasing and $(x_{n_k}) \to L$. The subsequence is increasing too, so $x_{n_k} \leq L$ for every k (Theorem 2.3.4 applied to $x_{n_k} \leq x_{n_j}$ for $j \geq k$). Given n, pick k with $n_k \geq n$; then $x_n \leq x_{n_k} \leq L$.

So (x_n) is increasing and bounded above, hence convergent by the Monotone Convergence Theorem (Theorem 2.4.2) — and its limit is L by Theorem 2.5.2. The decreasing case is identical with the inequalities reversed.

Problem 2.5.3. (a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1+a_2+a_3+a_4+a_5+\cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.
(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution. (a) The regrouped series has partial sums $t_k = s_{n_k}$, a subsequence of (s_n), so $(t_k) \to L$ by Theorem 2.5.2. Writing $n_0 = 0$ and $b_j = a_{n_{j-1}+1} + \cdots + a_{n_j}$ for the jth grouped term,

$$\begin{aligned} t_k &= \sum_{j=1}^{k} b_j = \sum_{j=1}^{k} \sum_{i=n_{j-1}+1}^{n_j} a_i \\ &= \sum_{i=1}^{n_k} a_i = s_{n_k}, \end{aligned}$$

the middle step being a regrouping of a finite sum. The indices satisfy $n_1 < n_2 < n_3 < \cdots$, so (s_{n_k}) is a subsequence of (s_n) in the sense of Definition 2.5.1 and Theorem 2.5.2 applies.
(b) The hypothesis of (a) — that (s_n) converges to some L — fails there. The example is $\sum_{n=1}^{\infty}(-1)^n$, whose two groupings

$$\begin{gathered} (-1+1) + (-1+1) + \cdots = 0, \\ -1 + (1-1) + (1-1) + \cdots = -1, \end{gathered}$$

are exactly the subsequences $(s_{2k}) = (0, 0, 0, \ldots)$ and $(s_{2k-1}) = (-1, -1, -1, \ldots)$ of the partial sums $(s_n) = (-1, 0, -1, 0, \ldots)$. Two subsequences with different limits force divergence by Theorem 2.5.2, so no L exists for the argument in (a) to transport.

Problem 2.5.4. The Bolzano–Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution. Bisect, keeping an upper bound on the right and a non-upper-bound on the left. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above; we produce $\sup A$.
Fix $x_0 \in A$ and an upper bound b_1 of A, and set $a_1 = x_0 - 1$, so that a_1 fails to be an upper bound of A (it is less than $x_0 \in A$). Put $I_1 = [a_1, b_1]$. Recursively, given $I_n = [a_n, b_n]$ with b_n an upper bound of A and a_n not one, let $m = (a_n + b_n)/2$ and set

$$I_{n+1} = \begin{cases} [a_n, m], & \text{if } m \text{ is an upper bound of } A, \\ [m, b_n], & \text{otherwise.} \end{cases}$$

In either case $I_{n+1} \subseteq I_n$, the right endpoint of I_{n+1} is an upper bound of A, the left endpoint is not one, and the lengths satisfy

$$\ell_n := b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \longrightarrow 0$$

by the assumption $(1/2^n) \to 0$ together with the Algebraic Limit Theorem (Theorem 2.3.3). The

Nested Interval Property (Theorem 1.4.1) supplies $s \in \bigcap_{n=1}^{\infty} I_n$. We claim $s = \sup A$.
(i) s is an upper bound. If some $a \in A$ had $a > s$, choose n with $\ell_n < a - s$. Since $s, b_n \in I_n$ and $b_n \geq s$,

$$b_n \leq s + \ell_n < s + (a - s) = a,$$

contradicting that b_n is an upper bound of A.
(ii) s is the least upper bound. If u were an upper bound with $u < s$, choose n with $\ell_n < s - u$. Since $a_n \leq s$ and $s - a_n \leq \ell_n$,

$$a_n \geq s - \ell_n > s - (s - u) = u.$$

But a_n is not an upper bound of A, so some $a \in A$ satisfies $a > a_n > u$, contradicting that u is an upper bound.
Hence $\sup A$ exists, which is the Axiom of Completeness.
Why the assumption is needed: steps (i) and (ii) require $\ell_n \to 0$, and $(1/2^n) \to 0$ is equivalent to the Archimedean Property (Theorem 1.4.2). Indeed $2^n \geq n$ turns the unboundedness of $\mathbf{N}$ into $1/2^n \to 0$, while conversely $1/2^n < \epsilon$ for large n exhibits the natural number 2^n exceeding $1/\epsilon$. Abbott derives Theorem 1.4.2 from AoC, so proving $(1/2^n) \to 0$ inside this argument would invoke the very axiom being established — and the hypothesis cannot be dropped, since NIP alone cannot prove $\mathbf{N}$ is unbounded in $\mathbf{R}$ (Section 2.6).

2.6 Exercises 2.5.5–2.6.2

Problem 2.5.5. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution. Suppose not. Then there is an $\epsilon_0 > 0$ for which no N works, i.e. $|a_n - a| \geq \epsilon_0$ for infinitely many n; list those indices as $n_1 < n_2 < n_3 < \cdots$, so

$$|a_{n_k} - a| \geq \epsilon_0 \qquad \text{for all } k.$$

The subsequence (a_{n_k}) inherits the bound on (a_n), so Bolzano–Weierstrass (Theorem 2.5.5) gives a convergent subsequence $(a_{n_{k_j}}) \to L$, and this is again a subsequence of (a_n) (the indices n_{k_j} are strictly increasing in j). By hypothesis $L = a$. But $|a_{n_{k_j}} - a| \to |L - a|$ (Theorem 2.3.3 and Exercise 2.3.10(b)), so the Order Limit Theorem (Theorem 2.3.4) applied to $|a_{n_{k_j}} - a| \geq \epsilon_0$ gives

$$0 = |L - a| = \lim_{j \to \infty} |a_{n_{k_j}} - a| \geq \epsilon_0 > 0,$$

a contradiction. Hence $(a_n) \to a$.

Problem 2.5.6. Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution. The limit is 1 for every $b > 0$, and 0 for $b = 0$.
For $b = 0$ the sequence is constantly 0, and for $b = 1$ it is constantly 1.
Case $b > 1$. The sequence $(b^{1/n})$ is decreasing and bounded below by 1: raising to the power $n(n+1)$ (strictly increasing on $[0, \infty)$) turns $b^{1/n} > b^{1/(n+1)}$ into $b^{n+1} > b^n$, which holds since $b > 1$, and $b^{1/n} \geq 1$ because $b \geq 1$. By the Monotone Convergence Theorem (Theorem 2.4.2) it converges to some l, and $l \geq 1$ by the Order Limit Theorem (Theorem 2.3.4).
Now identify l the way Example 2.5.3 identifies $\lim b^n$. The sequence $(b^{1/(2n)})_{n \geq 1}$ is the subsequence of $(b^{1/n})$ with indices $n_k = 2k$, so it converges to l by Theorem 2.5.2. On the other hand $b^{1/(2n)} = \sqrt{b^{1/n}}$, so Exercise 2.3.1(b) gives $b^{1/(2n)} \to \sqrt{l}$. Limits are unique (Theorem 2.2.7), hence

$$l = \sqrt{l} \implies l^2 = l \implies l \in \{0, 1\},$$

and $l \geq 1$ forces $l = 1$.

Case $0 < b < 1$. Then $1/b > 1$, so $(1/b)^{1/n} \to 1$ by the previous case, and the Algebraic Limit Theorem (Theorem 2.3.3) gives

$$b^{1/n} = \frac{1}{(1/b)^{1/n}} \longrightarrow \frac{1}{1} = 1$$

(the quotient rule applies since the limit 1 of the denominator is nonzero).

Problem 2.5.7. Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Solution. Both directions run off the identity $|b^n - 0| = |b|^n$.
($\Leftarrow$) Assume $-1 < b < 1$. If $b = 0$ the sequence is constantly 0. Otherwise $0 < |b| < 1$, so Example 2.5.3 gives $|b|^n \to 0$: for $\epsilon > 0$ there is N with $|b|^n < \epsilon$ whenever $n \geq N$. Since $|b^n - 0| = |b|^n$, the same N shows $(b^n) \to 0$ by Definition 2.2.3.
($\Rightarrow$) Contrapositive: assume $|b| \geq 1$. Then $|b|^n \geq 1$ for all n (induction: $|b|^1 \geq 1$, and $|b|^{n+1} = |b|^n|b| \geq 1 \cdot 1$), so

$$|b^n - 0| = |b|^n \geq 1 \qquad \text{for every } n,$$

and no N answers the challenge $\epsilon = 1$. Hence (b^n) does not converge to 0, and $\lim(b^n) = 0$ forces $|b| < 1$, i.e. $-1 < b < 1$.

Problem 2.5.8. Another way to prove the Bolzano–Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term.* Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.
(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.
(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano–Weierstrass Theorem.

Solution. (a) Four sequences, each listed with its peak set.

$$\begin{array}{rll}
\text{zero peaks:} & x_n = n, & \text{peaks: none,} \\
\text{one peak:} & 1,\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \ldots, & \text{peaks: } x_1, \\
\text{two peaks:} & 2,\ 1,\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \ldots, & \text{peaks: } x_1, x_2, \\
\text{infinitely many:} & x_n = \dfrac{(-1)^n}{n}, & \text{peaks: } x_2, x_4, x_6, \ldots
\end{array}$$

In the first, $x_{m+1} > x_m$ for every m, so nothing is a peak. In the second, $x_n = 1 - 1/n < 1 = x_1$ for $n \geq 2$ and the tail is strictly increasing, so x_1 is the only peak; the third prepends 2 to it, making x_1 and $x_2 = 1$ peaks and leaving that tail peak-free. In the fourth, $x_{2k} = 1/(2k) \geq x_n$ for all $n \geq 2k$ (later terms are negative or at most $1/(2k+2)$), so every even index is a peak, while the alternating sign rules out monotonicity.
(b) Claim: every sequence (x_n) contains a monotone subsequence. Split on the number of peak terms.
(i) Infinitely many peaks. Let $n_1 < n_2 < n_3 < \cdots$ list their indices. Since x_{n_k} is a peak and $n_{k+1} > n_k$,

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots,$$

a decreasing subsequence.
(ii) Finitely many peaks. Choose N larger than every peak index (take $N = 1$ if there are none) and set $n_1 = N$. Given $n_k \geq N$, the term x_{n_k} is not a peak, so some $n_{k+1} > n_k$ has $x_{n_{k+1}} > x_{n_k}$; and $n_{k+1} > N$, so the construction repeats. This yields

$$x_{n_1} < x_{n_2} < x_{n_3} < \cdots,$$

an increasing subsequence.

Now let (x_n) be bounded. By the claim it has a monotone subsequence (x_{n_k}), which inherits the bound, so (x_{n_k}) converges by the Monotone Convergence Theorem (Theorem 2.4.2). Every bounded sequence therefore has a convergent subsequence, which is the Bolzano–Weierstrass Theorem (Theorem 2.5.5).

Problem 2.5.9. Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano–Weierstrass Theorem using the Axiom of Completeness.)

Solution. For each $k \in \mathbf{N}$ the index set $\{n : s-1/k < a_n \leq s+1/k\}$ is infinite; picking $n_1 < n_2 < n_3 < \cdots$ out of these sets gives the subsequence.
First, s exists. Fix M with $|a_n| \leq M$ for all n. Then $-M-1 < a_n$ for every n, so $-M-1 \in S$ and $S \neq \emptyset$; and if $x \geq M$ then $a_n \leq M \leq x$ for every n, so $x \notin S$, making M an upper bound for S. The Axiom of Completeness supplies $s = \sup S$.
Two facts about s, for arbitrary $\epsilon > 0$:
(i) $s - \epsilon \in S$. Since $s-\epsilon$ is not an upper bound of S, there is $x \in S$ with $s - \epsilon < x$. That x satisfies $x < a_n$ for infinitely many n, and each such n has $s - \epsilon < x < a_n$. So

$$\{n \in \mathbf{N} : a_n > s - \epsilon\} \text{ is infinite.}$$

(ii) $s + \epsilon \notin S$, because $s + \epsilon > \sup S$. By the definition of S this says

$$\{n \in \mathbf{N} : a_n > s + \epsilon\} \text{ is finite.}$$

Subtracting the finite set in (ii) from the infinite set in (i),

$$E_\epsilon := \{n : s - \epsilon < a_n \leq s + \epsilon\}$$

is infinite for every $\epsilon > 0$.
Construct the indices inductively: let n_1 be any element of E_1, and having chosen $n_1 < \cdots < n_{k-1}$, use the infinitude of $E_{1/k}$ to pick $n_k \in E_{1/k}$ with $n_k > n_{k-1}$. Then (a_{n_k}) is a genuine subsequence and

$$|a_{n_k} - s| \leq \frac{1}{k} \qquad \text{for all } k.$$

Given $\epsilon > 0$, the Archimedean Property (Theorem 1.4.2) gives $K \in \mathbf{N}$ with $1/K < \epsilon$; for $k \geq K$ we get $|a_{n_k} - s| \leq 1/k \leq 1/K < \epsilon$. Hence $(a_{n_k}) \to s$.

Problem 2.6.1. Supply a proof for Theorem 2.6.2 (every convergent sequence is a Cauchy sequence).

Solution. Split the prescribed ϵ in half and insert the limit.
Assume $(x_n) \to x$ and let $\epsilon > 0$. By Definition 2.2.3 applied to $\epsilon/2$, there is $N \in \mathbf{N}$ with $|x_n - x| < \epsilon/2$ for all $n \geq N$. If $m, n \geq N$, the triangle inequality gives

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

This is precisely Definition 2.6.1, so (x_n) is Cauchy.

Problem 2.6.2. Give an example of each of the following, or argue that such a request is impossible.
(a) A Cauchy sequence that is not monotone.
(b) A Cauchy sequence with an unbounded subsequence.
(c) A divergent monotone sequence with a Cauchy subsequence.
(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. (a) $a_n = (-1)^n/n$. It converges to 0, hence is Cauchy by Theorem 2.6.2, and $a_1 = -1 < a_2 = 1/2 > a_3 = -1/3$, so it is neither increasing nor decreasing.
(b) Impossible. Lemma 2.6.3 says a Cauchy sequence (x_n) is bounded, say $|x_n| \le M$ for all n; every subsequence (x_{n_k}) satisfies the same bound $|x_{n_k}| \le M$.
(c) Impossible. Suppose (a_n) is monotone — say increasing — and has a Cauchy subsequence (a_{n_k}). By Lemma 2.6.3 there is M with $a_{n_k} \le M$ for all k. Given any n, choose k with $n_k \ge n$; monotonicity gives

$$a_n \le a_{n_k} \le M,$$

so (a_n) is bounded above, and bounded below by a_1; the Monotone Convergence Theorem (Theorem 2.4.2) makes it convergent, not divergent. If (a_n) is decreasing, apply the same argument to $(-a_n)$.
(d) Possible:

$$a_n = \begin{cases} n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The odd terms show (a_n) is unbounded, while the subsequence $a_2, a_4, a_6, \ldots$ is identically 0 and therefore Cauchy.

2.7 Exercises 2.6.3–2.7.2

Problem 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.
(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.
(b) Do the same for the product $(x_n y_n)$.

Solution. (a) Let $\epsilon > 0$. Definition 2.6.1 applied to each sequence with $\epsilon/2$ supplies N_1 and N_2; put $N = \max\{N_1, N_2\}$. For $m, n \ge N$,

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &\le |x_n - x_m| + |y_n - y_m| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

(b) Boundedness does the work. By Lemma 2.6.3 there is $M > 0$ with $|x_n| \le M$ and $|y_n| \le M$ for all n. Given $\epsilon > 0$, choose N so that both $|x_n - x_m| < \epsilon/(2M)$ and $|y_n - y_m| < \epsilon/(2M)$ whenever $m, n \ge N$. Then for such m, n, inserting the cross term $x_n y_m$ gives

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n(y_n - y_m) + y_m(x_n - x_m)| \\ &\le |x_n|\,|y_n - y_m| + |y_m|\,|x_n - x_m| \\ &< M \cdot \tfrac{\epsilon}{2M} + M \cdot \tfrac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Problem 2.6.4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.
(a) $c_n = |a_n - b_n|$
(b) $c_n = (-1)^n a_n$
(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution. (a) Cauchy. The reverse triangle inequality (Exercise 1.2.6 (d)) applied to $u = a_n - b_n$ and $v = a_m - b_m$ gives

$$\begin{aligned}\big||a_n - b_n| - |a_m - b_m|\big| &\le |(a_n - b_n) - (a_m - b_m)| \\ &\le |a_n - a_m| + |b_n - b_m|,\end{aligned}$$

so choosing N that makes each of the last two terms less than $\epsilon/2$ for $m, n \ge N$ makes $|c_n - c_m| < \epsilon$.
(b) Not Cauchy in general. Take $a_n = 1$ for all n, which is Cauchy; then $c_n = (-1)^n$ and $|c_{n+1} - c_n| = 2$ for every n, so no N works for $\epsilon = 2$.
(c) Not Cauchy in general. Take $a_n = (-1)^n/n$, which converges to 0 and so is Cauchy by Theorem 2.6.2. For even n we have $0 < a_n < 1$ and $[[a_n]] = 0$; for odd n we have $-1 \le a_n < 0$ and $[[a_n]] = -1$. Thus $|c_{n+1} - c_n| = 1$ for all n, and $\epsilon = 1$ admits no N.

Problem 2.6.5. Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \ge N$, then $|s_{n+1} - s_n| < \epsilon$.
Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.
(i) Pseudo-Cauchy sequences are bounded.
(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution. (ii) is the true one.
Proof of (ii). Let $\epsilon > 0$ and choose N_1, N_2 so that $|x_{n+1} - x_n| < \epsilon/2$ for $n \ge N_1$ and $|y_{n+1} - y_n| < \epsilon/2$ for $n \ge N_2$. For $n \ge N = \max\{N_1, N_2\}$,

$$\begin{aligned}|(x_{n+1} + y_{n+1}) - (x_n + y_n)| &\le |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

Counterexample to (i). Let $s_n = \sum_{k=1}^{n} 1/k$ be the partial sums of the harmonic series. Then

$$|s_{n+1} - s_n| = \frac{1}{n+1} < \epsilon \quad \text{whenever } n \ge N > 1/\epsilon,$$

such an N existing by the Archimedean Property (Theorem 1.4.2), so (s_n) is pseudo-Cauchy. But (s_n) is unbounded: Example 2.4.5 shows the harmonic series diverges, and since (s_n) is increasing, a bound would force convergence by the Monotone Convergence Theorem (Theorem 2.4.2).

Problem 2.6.6. Let us call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \ge N$ it follows that $a_n > a_m - \epsilon$.
(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.
(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.
(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? Give an example of a bounded, quasi-increasing sequence that does not converge, or prove that no such sequence exists.

Solution. (a) $a_n = (-1)^n/n$. For $n > m \ge N$,

$$a_n - a_m \ge -\frac{1}{n} - \frac{1}{m} > -\frac{2}{N},$$

so any $N > 2/\epsilon$ (Theorem 1.4.2) works. It is monotone on no tail because

$$a_{n+1} - a_n = (-1)^{n+1}\left(\tfrac{1}{n+1} + \tfrac{1}{n}\right)$$

changes sign at every n.

(b) $a_n = \sqrt{n} + (-1)^n n^{-1/4}$.
Quasi-increasing: for $n > m \geq N$ we have $\sqrt{n} - \sqrt{m} > 0$, hence

$$a_n - a_m > -n^{-1/4} - m^{-1/4} \geq -2N^{-1/4},$$

and any $N > (2/\epsilon)^4$ gives $2N^{-1/4} < \epsilon$, hence $a_n > a_m - \epsilon$.
Divergent: $a_n \geq \sqrt{n} - 1$, which is unbounded, so (a_n) cannot converge (Theorem 2.3.2).
Nowhere monotone: writing

$$a_{n+1} - a_n = (\sqrt{n+1} - \sqrt{n}) + (-1)^{n+1}\big[(n+1)^{-1/4} + n^{-1/4}\big],$$

the alternating bracket dominates, since for every $n \geq 1$

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \\ &< n^{-1/4} < (n+1)^{-1/4} + n^{-1/4},\end{aligned}$$

the middle step because $2n^{1/4} > 1$. Hence $a_{n+1} - a_n$ has the sign of $(-1)^{n+1}$, which alternates.
(c) Yes — every bounded quasi-increasing sequence converges.
Let (a_n) be bounded and quasi-increasing. By the Bolzano–Weierstrass Theorem (Theorem 2.5.5) there is a subsequence $(a_{n_k}) \to a$. Fix $\epsilon > 0$. Choose N from the quasi-increasing property for $\epsilon/3$, and then K with $n_K \geq N$ and $|a_{n_k} - a| < \epsilon/3$ for all $k \geq K$. Let $n \geq n_K$.
(i) Lower bound: if $n > n_K$ then $n > n_K \geq N$ gives $a_n > a_{n_K} - \epsilon/3 > a - 2\epsilon/3$.
(ii) Upper bound: choose $k \geq K$ with $n_k > n$ (possible since $n_k \to \infty$); then $n_k > n \geq n_K \geq N$ gives $a_{n_k} > a_n - \epsilon/3$, so

$$a_n < a_{n_k} + \tfrac{\epsilon}{3} < a + \tfrac{2\epsilon}{3}.$$

Case (ii) already covers $n = n_K$, and there the lower bound is $a_{n_K} > a - \epsilon/3$ directly. So $|a_n - a| < \epsilon$ for all $n \geq n_K$, i.e. $(a_n) \to a$.

Problem 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.
(a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.
(b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.
(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution. (a) Let (a_n) be increasing and bounded (the decreasing case follows by applying this one to $(-a_n)$). By BW (Theorem 2.5.5) there is a subsequence $(a_{n_k}) \to a$; we show $(a_n) \to a$.
First, $a_n \leq a$ for every n. Otherwise $a_m > a$ for some m, and since $a_{n_k} \geq a_m$ for every $n_k \geq m$, the Order Limit Theorem (Theorem 2.3.4 (iii), in the eventual form licensed by the remark on tails that follows it) gives $a = \lim a_{n_k} \geq a_m > a$, a contradiction.
Now let $\epsilon > 0$ and pick K with $|a_{n_K} - a| < \epsilon$. For $n \geq n_K$, monotonicity and the previous paragraph give

$$a - \epsilon < a_{n_K} \leq a_n \leq a < a + \epsilon,$$

so $|a_n - a| < \epsilon$. Every ϵ is handled by an index supplied by the subsequence itself, so no statement of the form $1/N < \epsilon$ — and hence no appeal to the Archimedean Property — occurs.
(b) Let (a_n) be bounded, say $a_n \in I_0 = [-M, M]$ for all n. Bisect I_0; at least one closed half contains

a_n for infinitely many indices n, and call it I_1. Repeating, we obtain nested intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots, \qquad |I_k| = \frac{2M}{2^k},$$

each containing infinitely many terms of the sequence. Choose $n_1 < n_2 < n_3 < \cdots$ with $a_{n_k} \in I_k$, which is possible precisely because each I_k captures infinitely many indices.
This subsequence is Cauchy: if $l \geq k \geq K$ then $a_{n_k}, a_{n_l} \in I_K$ by nesting, so

$$|a_{n_k} - a_{n_l}| \leq |I_K| = \frac{2M}{2^K}.$$

Given $\epsilon > 0$ we need a K with $2M/2^K < \epsilon$, i.e. with $2^K > 2M/\epsilon$; since $2^K \geq K$, it suffices that $\mathbf{N}$ contain an element exceeding $2M/\epsilon$. This last sentence is the implicit use of the Archimedean Property (Theorem 1.4.2): without the guarantee that $\mathbf{N}$ is unbounded in $\mathbf{R}$, the lengths $2M/2^k$ need not be eventually smaller than ϵ. With it, (a_{n_k}) is Cauchy, and the Cauchy Criterion (Theorem 2.6.4) makes it convergent.
(c) Because $\mathbf{Q}$ is a counterexample. The rationals form an ordered field in which the Archimedean Property holds (given $x = p/q > 0$ and $y = r/s$, the integer $n = |r|q + 1$ satisfies $nx > y$), but AoC fails there: the set $S = \{x \in \mathbf{Q} : x^2 < 2\}$ is nonempty and bounded above yet has no least upper bound in $\mathbf{Q}$: the trichotomy argument used to prove Theorem 1.4.5 rules out $\alpha^2 < 2$ and $\alpha^2 > 2$ for a least upper bound α of S, leaving $\alpha^2 = 2$, which no rational satisfies (Theorem 1.1.1). A derivation of AoC from the Archimedean Property together with only the field and order axioms would be valid verbatim in $\mathbf{Q}$, and would therefore prove a false statement.

Problem 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs. [Theorem 2.7.7 assumes (i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and (ii) $(a_n) \to 0$; together these force $a_n \geq 0$.]
(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.
(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).
(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution. All three proofs run off one estimate. *Lemma.* With $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$, where $a_1 \geq a_2 \geq \cdots \geq 0$ and $(a_n) \to 0$, for $n > m \geq 0$

$$0 \;\leq\; a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n \;\leq\; a_{m+1}.$$

Call the middle quantity T. Bracketing from the left,

$$T = (a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots,$$

a sum of nonnegative pairs, plus a final $+a_n \geq 0$ when $n - m$ is odd; hence $T \geq 0$. Bracketing after the first term,

$$T = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots,$$

where every bracket is nonnegative and a final $-a_n \leq 0$ appears when $n - m$ is even; hence $T \leq a_{m+1}$.
(a) Since $s_n - s_m = \pm(a_{m+1} - a_{m+2} + \cdots \pm a_n)$ for $n > m$, the Lemma gives

$$|s_n - s_m| \leq a_{m+1} \qquad (n > m).$$

Let $\epsilon > 0$. As $(a_n) \to 0$, choose N with $a_{N+1} < \epsilon$; monotonicity gives $a_{m+1} \leq a_{N+1}$ for $m \geq N$, so $n > m \geq N$ forces $|s_n - s_m| \leq a_{m+1} < \epsilon$. Thus (s_n) is Cauchy and converges by the Cauchy Criterion (Theorem 2.6.4).

(b) Set $I_n = [\,s_{2n},\, s_{2n-1}\,]$. These are genuine intervals and they nest:

$$\begin{aligned} s_{2n-1} - s_{2n} &= a_{2n} \geq 0, \\ s_{2n+2} - s_{2n} &= a_{2n+1} - a_{2n+2} \geq 0, \\ s_{2n+1} - s_{2n-1} &= a_{2n+1} - a_{2n} \leq 0, \end{aligned}$$

so $I_{n+1} \subseteq I_n$. By the Nested Interval Property (Theorem 1.4.1) there is some $L \in \bigcap_{n=1}^{\infty} I_n$. Fix n. For every $k \geq n$ the endpoints s_{2k}, s_{2k-1} of I_k lie in I_n, so every s_m with $m \geq 2n-1$ lies in I_n, as does L. Hence

$$|s_m - L| \leq \operatorname{length}(I_n) = a_{2n} \qquad (m \geq 2n-1).$$

Given $\epsilon > 0$, pick n with $a_{2n} < \epsilon$ and take $N = 2n-1$; then $(s_n) \to L$.
(c) The three displayed inequalities in (b) say (s_{2n}) is increasing and (s_{2n-1}) is decreasing, while $s_{2n} \leq s_{2n-1} \leq s_1 = a_1$ and $s_{2n-1} \geq s_{2n} \geq s_2$. So (s_{2n}) is increasing and bounded above, (s_{2n+1}) is decreasing and bounded below, and the Monotone Convergence Theorem (Theorem 2.4.2) yields

$$s_{2n} \to L, \qquad s_{2n+1} \to L'.$$

Since $s_{2n+1} - s_{2n} = a_{2n+1} \to 0$, the Algebraic Limit Theorem (Theorem 2.3.3) gives $L' = L$. The two subsequences exhaust (s_n), so taking N past both of their ϵ-thresholds gives $(s_n) \to L$; hence $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = L$.

Problem 2.7.2. Decide whether each of the following series converges or diverges:
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$
(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$
(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$
(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Solution. (a), (b) converge; (c), (d), (e) diverge.
(a) Converges. For all $n \in \mathbf{N}$,

$$0 \leq \frac{1}{2^n + n} \leq \frac{1}{2^n} = \left(\tfrac{1}{2}\right)^n,$$

and $\sum (1/2)^n$ is geometric with $|r| = 1/2 < 1$, hence convergent (Example 2.7.5). The Comparison Test (Theorem 2.7.4 (i)) applies.
(b) Converges. Since $|\sin(n)| \leq 1$,

$$0 \leq \left| \frac{\sin(n)}{n^2} \right| \leq \frac{1}{n^2},$$

and $\sum 1/n^2$ converges by Corollary 2.4.7 ($p = 2 > 1$). So $\sum |\sin(n)/n^2|$ converges by Theorem 2.7.4 (i), and the Absolute Convergence Test (Theorem 2.7.6) finishes it.
(c) Diverges. The nth term is $(-1)^{n+1}\dfrac{n+1}{2n}$ (check $n = 1$: $2/2 = 1$), and

$$\left| \frac{n+1}{2n} \right| = \frac{1}{2} + \frac{1}{2n} \longrightarrow \frac{1}{2} \neq 0,$$

so the terms do not tend to 0. By the contrapositive of Theorem 2.7.3, the series diverges.

(d) Diverges. The terms come in blocks of three with signs $+, +, -$, so for $m \in \mathbf{N}$

$$\begin{aligned} s_{3m} &= \sum_{k=0}^{m-1} \left(\frac{1}{3k+1} + \frac{1}{3k+2} - \frac{1}{3k+3} \right) \\ &\geq \sum_{k=0}^{m-1} \frac{1}{3k+1} \;\geq\; \frac{1}{3} \sum_{k=0}^{m-1} \frac{1}{k+1}, \end{aligned}$$

using $1/(3k+2) \geq 1/(3k+3)$ for the first inequality and $1/(3k+1) \geq 1/(3k+3)$ for the second. The harmonic partial sums are unbounded (Example 2.4.5), so (s_{3m}) is unbounded and (s_n) cannot converge, since convergent sequences are bounded (Theorem 2.3.2).
(e) Diverges. Pairing an odd term with the following even term,

$$\begin{aligned} s_{2m} &= \sum_{k=1}^{m} \left(\frac{1}{2k-1} - \frac{1}{(2k)^2} \right) \\ &\geq \frac{1}{2} \sum_{k=1}^{m} \frac{1}{k} \;-\; \sum_{k=1}^{\infty} \frac{1}{k^2}, \end{aligned}$$

because $1/(2k-1) \geq 1/(2k)$ and $\sum_{k=1}^{m} 1/(2k)^2 \leq \sum_{k=1}^{\infty} 1/k^2 =: M < \infty$ by Corollary 2.4.7. The right side tends to ∞, so (s_{2m}) is unbounded and (s_n) diverges (Theorem 2.3.2).

2.8 Exercises 2.7.3–2.7.9

Problem 2.7.3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.
(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.
[Theorem 2.7.4 assumes (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, and asserts (i) if $\sum_{k=1}^{\infty} b_k$ converges then so does $\sum_{k=1}^{\infty} a_k$, and (ii) if $\sum_{k=1}^{\infty} a_k$ diverges then so does $\sum_{k=1}^{\infty} b_k$.]

Solution. Part (ii) is the contrapositive of part (i), so each proof need only establish (i).
(a) Let $\epsilon > 0$. Since $\sum_{k=1}^{\infty} b_k$ converges, the Cauchy Criterion for Series (Theorem 2.7.2) supplies $N \in \mathbf{N}$ with

$$|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon \qquad \text{whenever } n > m \geq N.$$

For those same $n > m \geq N$, nonnegativity lets us drop the absolute values and the hypothesis $a_k \leq b_k$ gives

$$\begin{aligned} |a_{m+1} + \cdots + a_n| &= a_{m+1} + \cdots + a_n \\ &\leq b_{m+1} + \cdots + b_n \\ &= |b_{m+1} + \cdots + b_n| \;<\; \epsilon. \end{aligned}$$

So $\sum a_k$ satisfies the Cauchy Criterion for Series, and the sufficiency direction of Theorem 2.7.2 gives convergence.
(b) Write $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Because $a_k \geq 0$ and $b_k \geq 0$, both (s_n) and (t_n) are increasing. Assume $\sum b_k$ converges; then (t_n) converges, hence is bounded (Theorem 2.3.2), say $t_n \leq M$ for all n. Term-by-term comparison gives

$$s_n = \sum_{k=1}^{n} a_k \;\leq\; \sum_{k=1}^{n} b_k = t_n \;\leq\; M.$$

Thus (s_n) is increasing and bounded above, so it converges by the Monotone Convergence Theorem (Theorem 2.4.2); that is, $\sum a_k$ converges.

Problem 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).
(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.
(d) A sequence (x_n) satisfying $0 \le x_n \le 1/n$ where $\sum(-1)^n x_n$ diverges.

Solution. (a) Take $x_n = y_n = n^{-2/3}$. Then $\sum n^{-2/3}$ diverges and $\sum x_n y_n = \sum n^{-4/3}$ converges, both by Corollary 2.4.7 ($p = 2/3 \le 1$, $p = 4/3 > 1$).
(b) Take $x_n = (-1)^n/n$ and $y_n = (-1)^n$. Since $(1/n)$ decreases to 0, the Alternating Series Test (Theorem 2.7.7) applies to $\sum(-1)^{n+1}/n$, and $\sum x_n$ is its (-1)-multiple, so it converges too (Theorem 2.7.1 (i)). The sequence (y_n) is bounded by 1, and

$$\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, which diverges (Example 2.4.5).
(c) Impossible. Writing $y_n = (x_n + y_n) + (-1)x_n$, the Algebraic Limit Theorem for Series (Theorem 2.7.1, parts (i) and (ii)) gives

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty}(x_n + y_n) - \sum_{n=1}^{\infty} x_n,$$

so $\sum y_n$ converges whenever the other two do.
(d) Take

$$x_n = \begin{cases} 1/n, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

which satisfies $0 \le x_n \le 1/n$. All odd terms of $\sum(-1)^n x_n$ vanish and all even terms are $+1/n$, so

$$s_{2m} = \sum_{k=1}^{m} \frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{m} \frac{1}{k},$$

which is unbounded (Example 2.4.5); hence (s_n) diverges by Theorem 2.3.2.

Problem 2.7.5. Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7. [Corollary 2.4.7 states: the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.]

Solution. Condense, then read off a geometric series.
(i) $p \le 0$. Then $1/n^p = n^{-p} \ge 1$ for every $n \in \mathbf{N}$, so the terms do not converge to 0 and the series diverges by the contrapositive of Theorem 2.7.3. This is consistent with the claim, since such p are not > 1.
(ii) $p > 0$. Set $b_n = 1/n^p$. Then $b_n > 0$ and (b_n) is decreasing, so the Cauchy Condensation Test (Theorem 2.4.6) applies: $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Now

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=0}^{\infty} \left(2^{1-p}\right)^n,$$

a geometric series with $a = 1$ and $r = 2^{1-p} > 0$. By Example 2.7.5 it converges if and only if $|r| < 1$, and since $x \mapsto 2^x$ is strictly increasing,

$$2^{1-p} < 1 = 2^0 \iff 1 - p < 0 \iff p > 1.$$

Combining (i) and (ii), $\sum_{n=1}^{\infty} 1/n^p$ converges precisely when $p > 1$.

Problem 2.7.6. Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:
(a) If (a_n) is bounded, then $\sum a_n$ subverges.
(b) All convergent series are subvergent.
(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.
(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution. (a) false, (b) true, (c) true, (d) false.
(a) False. Take $a_n = 1$, a bounded sequence. Then $s_n = n$, and every subsequence $(s_{n_k}) = (n_k)$ is unbounded, hence divergent (Theorem 2.3.2). So $\sum a_n$ does not subverge.
(b) True. If $\sum a_n$ converges, then (s_n) converges, and (s_n) is a subsequence of itself (Definition 2.5.1 with $n_k = k$).
(c) True. Let $t_n = |a_1| + \cdots + |a_n|$, an increasing sequence, and suppose some subsequence (t_{n_k}) converges; being convergent it is bounded, say $t_{n_k} \le M$ for all k. Given any n, choose k with $n_k \ge n$; then $t_n \le t_{n_k} \le M$. So (t_n) is increasing and bounded, hence convergent by the Monotone Convergence Theorem (Theorem 2.4.2). Thus $\sum |a_n|$ converges, so $\sum a_n$ converges by the Absolute Convergence Test (Theorem 2.7.6), and therefore subverges by (b).
(d) False. Take

$$(a_n) = 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \ldots, \qquad \text{i.e. } a_{2k-1} = k,\ a_{2k} = -k.$$

Then $s_{2k} = 0$ for every k, so the subsequence (s_{2k}) converges and $\sum a_n$ subverges. But $|a_n| \to \infty$, so every subsequence of (a_n) is unbounded and hence divergent (Theorem 2.3.2).

Problem 2.7.7. (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.
(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution. (a) Compare with the harmonic series. Since $na_n > 0$ for all n, the Order Limit Theorem (Theorem 2.3.4) gives $l \ge 0$, and $l \neq 0$ forces $l > 0$. Apply the definition of $\lim(na_n) = l$ with $\epsilon = l/2$: there is $N \in \mathbf{N}$ such that

$$na_n > l - \tfrac{l}{2} = \tfrac{l}{2},$$
$$\text{hence} \qquad a_n \ge \frac{l}{2} \cdot \frac{1}{n} \ge 0,$$

for all $n \ge N$. The series $\sum (l/2)(1/n)$ diverges, since $\sum 1/n$ does (Corollary 2.4.7 with $p = 1$) and a nonzero constant multiple of a divergent series diverges (Theorem 2.7.1 (i), applied to the factor $2/l$). By the Comparison Test (Theorem 2.7.4 (ii)) applied to the tails from index N on, $\sum_{n \ge N} a_n$ diverges, and since convergence is unaffected by the finitely many terms $a_1, \ldots, a_{N-1}$, the series $\sum a_n$ diverges.
(b) Compare with $\sum 1/n^2$. The sequence $(n^2 a_n)$ converges by hypothesis, hence is bounded (Theorem 2.3.2): there is $M > 0$ with $n^2 a_n \le M$ for all n. Therefore

$$0 < a_n \le \frac{M}{n^2} \qquad \text{for all } n \in \mathbf{N}.$$

Now $\sum 1/n^2$ converges by Corollary 2.4.7 ($p = 2 > 1$), so $\sum M/n^2$ converges by Theorem 2.7.1 (i), and the Comparison Test (Theorem 2.7.4 (i)) gives convergence of $\sum a_n$.

Problem 2.7.8. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.
(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.
(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.
(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution. (a) True, (b) false, (c) true.
(a) Because $\sum |a_n|$ converges, Theorem 2.7.3 gives $(|a_n|) \to 0$, so there is an N with $|a_n| < 1$ for all $n \geq N$. For such n,

$$|a_n^2| \;=\; |a_n|^2 \;\leq\; |a_n|,$$

and the Comparison Test (Theorem 2.7.4) applied to the tail $\sum_{n\geq N} |a_n^2|$ finishes it; the finitely many terms before N do not affect convergence.
(b) Take

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n}}.$$

Then $\sum a_n$ converges by the Alternating Series Test (Theorem 2.7.7), since $1/\sqrt{n}$ decreases to 0, and $(b_n) \to 0$ converges. But

$$\sum_{n=1}^{\infty} a_n b_n \;=\; \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges (Corollary 2.4.7).
(c) Suppose $\sum n^2 a_n$ converged. Then $(n^2 a_n) \to 0$ by Theorem 2.7.3, so there is an N with $|n^2 a_n| \leq 1$, i.e.

$$|a_n| \;\leq\; \frac{1}{n^2} \qquad \text{for all } n \geq N.$$

Since $\sum 1/n^2$ converges (Corollary 2.4.7), the Comparison Test (Theorem 2.7.4) makes $\sum |a_n|$ converge, contradicting the hypothesis that $\sum a_n$ converges only conditionally (Definition 2.7.8). Hence $\sum n^2 a_n$ diverges.

Problem 2.7.9. (Ratio Test). Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.
(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.
(b) Why does $|a_N| \sum (r')^n$ converge?
(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution. (a) Apply the definition of the limit with $\epsilon = r' - r > 0$: there is an $N \in \mathbf{N}$ such that

$$\begin{aligned} n \geq N \implies & \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| < r' - r \\ \implies & \left| \frac{a_{n+1}}{a_n} \right| < r', \end{aligned}$$

and multiplying through by $|a_n| > 0$ gives $|a_{n+1}| \leq |a_n| r'$ for all $n \geq N$.
(b) Note $0 \leq r < r' < 1$, since r is a limit of absolute values. So $\sum (r')^n$ is a geometric series with ratio of absolute value less than 1, hence convergent (Example 2.7.5), and multiplying by the constant $|a_N|$ preserves convergence by Theorem 2.7.1(i).
(c) Iterating the inequality in (a) from $n = N$ upward gives, by induction,

$$|a_{N+k}| \;\leq\; |a_N| (r')^k \qquad \text{for all } k \geq 0.$$

The series $\sum_{k=0}^{\infty} |a_N| (r')^k$ converges by (b), so the Comparison Test (Theorem 2.7.4), whose positivity hypothesis holds since all terms here are nonnegative, shows $\sum_{k=0}^{\infty} |a_{N+k}|$ converges. Adding back the finitely many terms $|a_1|, \ldots, |a_{N-1}|$ changes only a partial sum, so $\sum_{n=1}^{\infty} |a_n|$ converges. By the Absolute Convergence Test (Theorem 2.7.6), $\sum a_n$ converges.

2.9 Exercises 2.7.10–2.8.2

Problem 2.7.10. (Infinite Products). Review Exercise 2.4.10 about infinite products and then answer the following questions:
(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?
(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?
(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Solution. (a) Yes. The n-th factor is $1+2^{-(n-1)}$, so the product is $\prod_{n=1}^{\infty}(1+a_n)$ with $a_n = 2^{-(n-1)} \geq 0$ and

$$\sum_{n=1}^{\infty} 2^{-(n-1)} \;=\; 2 < \infty$$

(a geometric series, Example 2.7.5). By Exercise 2.4.10(b) the product converges.
(b) It converges because its partial products

$$p_n \;=\; \prod_{k=1}^{n} \frac{2k-1}{2k}$$

are decreasing (every factor lies in $(0,1)$) and bounded below by 0, so the Monotone Convergence Theorem (Theorem 2.4.2) applies. The limit is zero. Indeed,

$$\frac{1}{p_n} \;=\; \prod_{k=1}^{n} \frac{2k}{2k-1} \;=\; \prod_{k=1}^{n}\left(1+\frac{1}{2k-1}\right),$$

and $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ diverges, since $\frac{1}{2k-1} \geq \frac{1}{2k}$ and the harmonic series diverges (Corollary 2.4.7). So Exercise 2.4.10(b) says the product $\prod(1+\frac{1}{2k-1})$ does not converge; being an increasing sequence, $(1/p_n)$ must therefore be unbounded, and hence $p_n \to 0$.
(c) The n-th factor is

$$\frac{(2n)(2n)}{(2n-1)(2n+1)} \;=\; \frac{4n^2}{4n^2-1} \;=\; 1+\frac{1}{4n^2-1},$$

so again the product has the form $\prod(1+a_n)$ with $a_n \geq 0$. Since $4n^2-1 \geq 3n^2$ for every $n \geq 1$,

$$0 \;\leq\; \frac{1}{4n^2-1} \;\leq\; \frac{1}{3n^2},$$

and $\sum 1/n^2$ converges by Corollary 2.4.7, so $\sum a_n$ converges by the Comparison Test (Theorem 2.7.4). Exercise 2.4.10(b) then gives convergence of the Wallis product.

Problem 2.7.11. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution. Take

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 1/n^2 & n \text{ even} \end{cases} \qquad b_n = \begin{cases} 1/n^2 & n \text{ odd} \\ 1 & n \text{ even.} \end{cases}$$

Neither (a_n) nor (b_n) tends to 0, so both series diverge by Theorem 2.7.3, while $\min\{a_n, b_n\} = 1/n^2$

and $\sum 1/n^2$ converges (Corollary 2.4.7).
For the strictly positive decreasing version, interleave long blocks. Define $n_1 = 1$ and $n_{k+1} = 2^{n_k^2} n_k$, and let $I_k = \{n \in \mathbf{N} : n_k \le n < n_{k+1}\}$, so that $\mathbf{N}$ is partitioned into the blocks $I_1, I_2, I_3, \ldots$. Put

$$c_n = \frac{1}{n^2}, \qquad d_n = \frac{1}{n_k\, n} \quad \text{for } n \in I_k,$$

and set, for $n \in I_k$,

$$a_n = \begin{cases} d_n & k \text{ odd} \\ c_n & k \text{ even,} \end{cases} \qquad b_n = \begin{cases} c_n & k \text{ odd} \\ d_n & k \text{ even.} \end{cases}$$

The minimum. For $n \in I_k$ we have $n \ge n_k$, hence $n_k n \le n^2$ and so $c_n \le d_n$. Therefore $\min\{a_n, b_n\} = c_n = 1/n^2$ for every n, and $\sum \min\{a_n, b_n\}$ converges by Corollary 2.4.7.
Strictly decreasing. Both c_n and d_n are strictly decreasing in n within a block. At the left endpoint of a block the two agree,

$$d_{n_{k+1}} = \frac{1}{n_{k+1} \cdot n_{k+1}} = c_{n_{k+1}},$$

so whichever branch is used, $a_{n_{k+1}} = b_{n_{k+1}} = 1/n_{k+1}^2$. The preceding term is either $c_{n_{k+1}-1} = 1/(n_{k+1}-1)^2$ or $d_{n_{k+1}-1} = 1/\big(n_k(n_{k+1}-1)\big)$, and both exceed $1/n_{k+1}^2$ because $(n_{k+1}-1)^2 < n_{k+1}^2$ and $n_k(n_{k+1}-1) < n_{k+1}^2$. So (a_n) and (b_n) are strictly decreasing and strictly positive.
Divergence. Write $m_k = n_k^2$, so $n_{k+1} = 2^{m_k} n_k$. Splitting I_k into the m_k ranges $[2^j n_k, 2^{j+1} n_k)$, each of which has $2^j n_k$ terms every one of which exceeds $1/(2^{j+1} n_k)$, gives

$$\sum_{n \in I_k} \frac{1}{n} > \frac{m_k}{2},$$
$$\sum_{n \in I_k} d_n = \frac{1}{n_k} \sum_{n \in I_k} \frac{1}{n} > \frac{n_k}{2}.$$

Since all terms are positive, the partial sums of $\sum a_n$ exceed $n_k/2$ for every odd k and the partial sums of $\sum b_n$ exceed $n_k/2$ for every even k; both are unbounded, so both series diverge.

Problem 2.7.12. (Summation-by-parts). Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Solution. Substitute $x_j = s_j - s_{j-1}$ and shift the index in the second sum:

$$\begin{aligned} \sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\ &= \sum_{j=m}^{n} s_j y_j - \sum_{j=m-1}^{n-1} s_j y_{j+1}. \end{aligned}$$

The second sum runs over $j = m-1, \ldots, n-1$; peel off its $j = m-1$ term and add in the $j = n$ term that is missing, which costs $-s_{m-1} y_m + s_n y_{n+1}$:

$$\begin{aligned} \sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} s_j y_j - \left(\sum_{j=m}^{n} s_j y_{j+1} + s_{m-1} y_m - s_n y_{n+1} \right) \\ &= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}), \end{aligned}$$

which is the asserted identity.

Problem 2.7.13. (Abel's Test). Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.
(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.
(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution. (a) Put $m = 1$ in the identity of Exercise 2.7.12. The boundary term becomes $-s_{m-1}y_m = -s_0 y_1 = 0$, since $s_0 = 0$ by definition, leaving

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}).$$

(b) Because $\sum x_k$ converges, the sequence (s_n) of partial sums converges and is therefore bounded (Theorem 2.3.2): fix $M > 0$ with $|s_k| \leq M$ for all k. Monotonicity of (y_k) makes $y_k - y_{k+1} \geq 0$, so

$$\left|s_k(y_k - y_{k+1})\right| \leq M(y_k - y_{k+1}) \qquad (k \in \mathbf{N}),$$

and the majorant telescopes:

$$\sum_{k=1}^{n} M(y_k - y_{k+1}) = M(y_1 - y_{n+1}).$$

The sequence (y_k) is decreasing and bounded below by 0, so it converges to some $y \geq 0$ by the Monotone Convergence Theorem (Theorem 2.4.2); hence these partial sums converge to $M(y_1 - y)$, i.e. $\sum M(y_k - y_{k+1})$ converges. Both sides being nonnegative, the Comparison Test (Theorem 2.7.4) gives convergence of $\sum |s_k(y_k - y_{k+1})|$, and then the Absolute Convergence Test (Theorem 2.7.6) gives convergence of $\sum s_k(y_k - y_{k+1})$ itself.
Now let $n \to \infty$ in (a). The first term satisfies $s_n y_{n+1} \to sy$, where $s = \lim s_n$, by the Algebraic Limit Theorem (Theorem 2.3.3), and the second term converges by the previous paragraph. Hence the partial sums of $\sum x_k y_k$ converge, which is Abel's Test.

Problem 2.7.14. (Dirichlet's Test). Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.
(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.
(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution. (a) The trade is one hypothesis for another: Abel demands that (s_n) actually converge but lets (y_k) decrease to any limit $y \geq 0$; Dirichlet demands only that (s_n) be bounded but insists that $y_k \to 0$. Since convergence implies boundedness (Theorem 2.3.2), Dirichlet weakens the hypothesis on (x_k) and strengthens the one on (y_k), so neither test contains the other.

The proof runs through the same identity from Exercise 2.7.13(a),

$$\sum_{k=1}^{n} x_k y_k \;=\; s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

and the estimate on the sum is unchanged, since it only ever used boundedness. Fix $M > 0$ with $|s_k| \le M$ for all k, now by hypothesis. Since (y_k) is decreasing, $y_k - y_{k+1} \ge 0$ and

$$\begin{aligned} \big|s_k(y_k - y_{k+1})\big| \;&\le\; M(y_k - y_{k+1}), \\ \sum_{k=1}^{n} M(y_k - y_{k+1}) &= M(y_1 - y_{n+1}) \to M y_1, \end{aligned}$$

using $y_k \to 0$. So the majorizing series converges, the Comparison Test (Theorem 2.7.4) makes $\sum |s_k(y_k - y_{k+1})|$ converge, and the Absolute Convergence Test (Theorem 2.7.6) makes $\sum s_k(y_k - y_{k+1})$ converge.

The one point that changes is the boundary term. Abel's argument used $\lim s_n$; here (s_n) need not converge, but instead

$$0 \;\le\; |s_n y_{n+1}| \;\le\; M\, y_{n+1} \;\longrightarrow\; 0,$$

so $s_n y_{n+1} \to 0$ by the Squeeze Theorem (Exercise 2.3.3). Letting $n \to \infty$ in the identity, the partial sums of $\sum x_k y_k$ converge, proving Dirichlet's Test.

(b) Let (a_n) satisfy the hypotheses of Theorem 2.7.7: $a_1 \ge a_2 \ge a_3 \ge \cdots$ and $(a_n) \to 0$ (so $a_n \ge 0$). Apply Dirichlet's Test with

$$x_k = (-1)^{k+1}, \qquad y_k = a_k.$$

The partial sums of $\sum x_k$ are

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even}, \end{cases}$$

hence bounded by 1, and (y_k) decreases to 0. Dirichlet's Test therefore gives convergence of

$$\sum_{k=1}^{\infty} x_k y_k \;=\; \sum_{k=1}^{\infty} (-1)^{k+1} a_k,$$

which is exactly the Alternating Series Test.

Problem 2.8.1. Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Here $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$, and the array of Section 2.1 is

$$a_{ij} = \frac{1}{2^{\,j-i}} \text{ if } j > i, \qquad a_{ij} = -1 \text{ if } j = i, \qquad a_{ij} = 0 \text{ if } j < i,$$

i.e.

$$\begin{bmatrix} -1 & 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 0 & -1 & 1/2 & 1/4 & 1/8 & \cdots \\ 0 & 0 & -1 & 1/2 & 1/4 & \cdots \\ 0 & 0 & 0 & -1 & 1/2 & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Section 2.1 computed the two iterated sums to be

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = 0 \qquad \text{and} \qquad \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = -2.$$

Solution. $\lim_{n\to\infty} s_{nn} = -2$: the square limit agrees with the column-first iterated value -2 and differs from the row-first value 0.

Fix n and sum row $i \le n$ across the first n columns. The entries left of the diagonal vanish, the diagonal entry is -1, and the remaining entries form a finite geometric sum:

$$\sum_{j=1}^{n} a_{ij} = -1 + \sum_{j=i+1}^{n} \frac{1}{2^{\,j-i}} = -1 + \sum_{k=1}^{n-i} \frac{1}{2^k}$$
$$= -1 + \left(1 - \frac{1}{2^{\,n-i}}\right) = -\frac{1}{2^{\,n-i}}.$$

Summing these n row-sums,

$$s_{nn} = -\sum_{i=1}^{n} \frac{1}{2^{\,n-i}} = -\frac{1}{2^n}\sum_{i=1}^{n} 2^i = -\frac{2^{n+1}-2}{2^n}$$
$$= -2 + \frac{1}{2^{\,n-1}}.$$

Hence $\lim_{n\to\infty} s_{nn} = -2$ by the Algebraic Limit Theorem (Theorem 2.3.3), since $(1/2^{\,n-1}) \to 0$.

Problem 2.8.2. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Solution. Each row sums absolutely, and the row sums are dominated by the b_i.

Fix i. Since $\sum_{j=1}^{\infty} |a_{ij}|$ converges, the Absolute Convergence Test (Theorem 2.7.6) gives that $\sum_{j=1}^{\infty} a_{ij}$ converges; call its value r_i, so that the inner series in the second display is meaningful for every i. For each n the triangle inequality gives

$$\left|\sum_{j=1}^{n} a_{ij}\right| \le \sum_{j=1}^{n} |a_{ij}| \le b_i,$$

and letting $n \to \infty$ the Order Limit Theorem (Theorem 2.3.4) yields

$$|r_i| \le b_i \qquad \text{for every } i \in \mathbf{N}.$$

Now $\sum_{i=1}^{\infty} b_i$ converges with $0 \le |r_i| \le b_i$, so the Comparison Test (Theorem 2.7.4) shows $\sum_{i=1}^{\infty} |r_i|$ converges, and a second application of Theorem 2.7.6 shows $\sum_{i=1}^{\infty} r_i$ converges. Since $r_i = \sum_{j=1}^{\infty} a_{ij}$, this is precisely the statement that the iterated series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges.

2.10 Exercises 2.8.3–2.8.7

Problem 2.8.3. This exercise is a step in the proof of Theorem 2.8.1, where $\{a_{ij} : i, j \in \mathbf{N}\}$ is a doubly indexed array whose iterated series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ converges, $b_i = \sum_{j=1}^{\infty}|a_{ij}|$, and the rectangular partial sums are

$$s_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}, \qquad t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|.$$

(a) Prove that (t_{nn}) converges.
(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

Solution. (a) (t_{nn}) is increasing and bounded above by $\sum_{i=1}^{\infty} b_i$.
Passing from t_{nn} to $t_{n+1,n+1}$ only adds terms $|a_{ij}| \geq 0$, so (t_{nn}) is increasing. For the bound, each inner sum is a partial sum of the convergent nonnegative series defining b_i, so $\sum_{j=1}^{n}|a_{ij}| \leq b_i$, whence

$$t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}| \leq \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{\infty} b_i,$$

the last step because (b_i) is nonnegative and $\sum_{i=1}^{\infty} b_i$ converges by hypothesis. The Monotone Convergence Theorem (Theorem 2.4.2) now gives convergence of (t_{nn}).
(b) Write $Q_n = \{(i,j) \in \mathbf{N}\times\mathbf{N} : 1 \leq i \leq n,\ 1 \leq j \leq n\}$. For $n > m$ the terms of s_{nn} not appearing in s_{mm} are exactly those indexed by the L-shaped region $Q_n \setminus Q_m$, so

$$\begin{aligned}|s_{nn} - s_{mm}| &= \left|\sum_{(i,j)\in Q_n\setminus Q_m} a_{ij}\right| \leq \sum_{(i,j)\in Q_n\setminus Q_m} |a_{ij}| \\ &= t_{nn} - t_{mm} = |t_{nn} - t_{mm}|,\end{aligned}$$

the last equality because (t_{nn}) is increasing. By (a) and Theorem 2.6.2, (t_{nn}) is a Cauchy sequence: given $\epsilon > 0$ there is an $N \in \mathbf{N}$ with $|t_{nn} - t_{mm}| < \epsilon$ whenever $n, m \geq N$. The displayed inequality then gives $|s_{nn} - s_{mm}| < \epsilon$ for all $n, m \geq N$, so (s_{nn}) is Cauchy and hence converges by the Cauchy Criterion (Theorem 2.6.4).

Problem 2.8.4. (Continuing the proof of Theorem 2.8.1. Here $S = \lim_{n\to\infty} s_{nn}$, which exists by Exercise 2.8.3, and $B = \sup\{t_{mn} : m, n \in \mathbf{N}\}$.)
(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies

$$B - \frac{\epsilon}{2} < t_{mn} \leq B.$$

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Solution. (a) Take $N_1 = \max\{m_0, n_0\}$, where $t_{m_0 n_0} > B - \epsilon/2$.
Such a pair exists because $B - \epsilon/2 < B$, so $B - \epsilon/2$ is not an upper bound for $\{t_{mn}\}$ (Lemma 1.3.8). Since all the $|a_{ij}|$ are nonnegative, t_{mn} is increasing in each index separately; hence for $m, n \geq N_1$ we have $m \geq m_0$ and $n \geq n_0$, so

$$B - \frac{\epsilon}{2} < t_{m_0 n_0} \leq t_{mn} \leq B,$$

the final inequality because B is an upper bound. (That B exists at all is the Axiom of Completeness

applied to $\{t_{mn}\}$, which is bounded above by $\sum_{i=1}^{\infty} b_i$ as in Exercise 2.8.3(a).)
(b) Choose $N \geq N_1$ with $|s_{pp} - S| < \epsilon/2$ for all $p \geq N$; this N works.
Let $m, n \geq N$ and set $p = \max\{m, n\}$. Writing $R_{mn} = \{(i,j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $Q_p = R_{pp}$, we have the nesting

$$Q_{N_1} \subseteq R_{mn} \subseteq Q_p,$$

the first inclusion because $m, n \geq N \geq N_1$. Therefore $Q_p \setminus R_{mn} \subseteq Q_p \setminus Q_{N_1}$, and since $p \geq N_1$,

$$\begin{aligned} |s_{pp} - s_{mn}| &\leq \sum_{(i,j)\in Q_p\setminus R_{mn}} |a_{ij}| \leq \sum_{(i,j)\in Q_p\setminus Q_{N_1}} |a_{ij}| \\ &= t_{pp} - t_{N_1N_1} < B - \left(B - \frac{\epsilon}{2}\right) = \frac{\epsilon}{2}, \end{aligned}$$

using part (a) on both $t_{pp} \leq B$ and $t_{N_1N_1} > B - \epsilon/2$. Since $p \geq N$, the triangle inequality finishes it:

$$\begin{aligned} |s_{mn} - S| &\leq |s_{mn} - s_{pp}| + |s_{pp} - S| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Problem 2.8.5. (Completing the proof of Theorem 2.8.1. For each fixed row i the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to a real number r_i; $\epsilon > 0$ and N are as in Exercise 2.8.4.)
(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.
(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Solution. (a) Fix $m \geq N$ and let $n \to \infty$ in $|s_{mn} - S| < \epsilon$.
Written out by rows,

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj},$$

a sum of exactly m sequences in n, the ith of which converges to r_i. The Algebraic Limit Theorem (Theorem 2.3.3), applied $m-1$ times, gives

$$\lim_{n\to\infty} s_{mn} = r_1 + r_2 + \cdots + r_m.$$

By Exercise 2.8.4(b), $-\epsilon < s_{mn} - S < \epsilon$ for every $n \geq N$, so the Order Limit Theorem (Theorem 2.3.4) applied to the sequence $(s_{mn} - S)_{n=N}^{\infty}$ yields

$$-\epsilon \leq (r_1 + \cdots + r_m) - S \leq \epsilon,$$

which is the assertion.
To conclude, let $S_m = r_1 + \cdots + r_m$ denote the mth partial sum of $\sum_{i=1}^{\infty} r_i$. Given an arbitrary $\alpha > 0$, run Exercise 2.8.4 with $\epsilon = \alpha/2$ to produce an N for which $m \geq N$ implies $|S_m - S| \leq \alpha/2 < \alpha$. Thus $(S_m) \to S$, i.e.

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} r_i = S.$$

(b) The columns converge absolutely, and the estimate of Exercise 2.8.4(b) is symmetric in m and n. Fix j. The partial sums $\sum_{i=1}^{m} |a_{ij}| \leq t_{mj} \leq B$ are increasing in m and bounded, so $\sum_{i=1}^{\infty} |a_{ij}|$ converges by the Monotone Convergence Theorem (Theorem 2.4.2) and $\sum_{i=1}^{\infty} a_{ij}$ converges to some c_j by Theorem 2.7.6. Now grouping the finite sum s_{mn} by columns instead of rows,

$$s_{mn} = \sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in},$$

so for fixed n the Algebraic Limit Theorem gives $\lim_{m\to\infty} s_{mn} = c_1 + \cdots + c_n$. Fixing $n \geq N$ and letting $m \to \infty$ in $-\epsilon < s_{mn} - S < \epsilon$, the Order Limit Theorem gives

$$|(c_1 + c_2 + \cdots + c_n) - S| \leq \epsilon \qquad (n \geq N),$$

and the same halving of ϵ as in (a) gives $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} c_j = S$.

Problem 2.8.6. A third common way of computing a double summation is to sum along the diagonals where $i + j$ equals a constant: given a doubly indexed array $\{a_{ij} : i, j \in \mathbf{N}\}$, set

$$d_2 = a_{11}, \quad d_3 = a_{12} + a_{21}, \quad d_4 = a_{13} + a_{22} + a_{31},$$

and in general

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$

(a) Assuming the hypothesis – and hence the conclusion – of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.
(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.
(The hypothesis of Theorem 2.8.1 is that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges; as above, $s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$, $t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$, and $B = \sup\{t_{mn} : m, n \in \mathbf{N}\}$.)

Solution. (a) The partial sums of $\sum |d_k|$ are bounded by $B = \sup\{t_{mn}\}$.
Write $T_K = \{(i, j) \in \mathbf{N} \times \mathbf{N} : i + j \leq K\}$, the Kth triangle; the terms making up $d_2, d_3, \ldots, d_K$ are precisely the a_{ij} with $(i, j) \in T_K$, each occurring once. Since $i + j \leq K$ with $i, j \geq 1$ forces $i \leq K - 1$ and $j \leq K - 1$, we have $T_K \subseteq Q_K$, and therefore

$$\sum_{k=2}^{K} |d_k| \leq \sum_{k=2}^{K} \sum_{i=1}^{k-1} |a_{i,k-i}| = \sum_{(i,j)\in T_K} |a_{ij}| \leq t_{KK} \leq B.$$

These partial sums are increasing and bounded above, so $\sum_{k=2}^{\infty} |d_k|$ converges by the Monotone Convergence Theorem (Theorem 2.4.2); that is, $\sum_{k=2}^{\infty} d_k$ converges absolutely, and in particular converges (Theorem 2.7.6).
(b) The triangle T_K traps the square $Q_{\lfloor K/2 \rfloor}$, so $D_K = \sum_{k=2}^{K} d_k$ has the same limit as s_{KK}.
Let $\epsilon > 0$ and take N_1 as in Exercise 2.8.4(a), so that $m, n \geq N_1$ implies $B - \epsilon/2 < t_{mn} \leq B$. Put $n_K = \lfloor K/2 \rfloor$. If $(i, j) \in Q_{n_K}$ then $i + j \leq 2n_K \leq K$, so

$$Q_{n_K} \subseteq T_K \subseteq Q_K,$$

and hence $Q_K \setminus T_K \subseteq Q_K \setminus Q_{n_K}$. For every $K \geq 2N_1$ we have $n_K \geq N_1$, and since $D_K = \sum_{(i,j)\in T_K} a_{ij}$ while $s_{KK} = \sum_{(i,j)\in Q_K} a_{ij}$,

$$\begin{aligned} |D_K - s_{KK}| &\leq \sum_{(i,j)\in Q_K \setminus T_K} |a_{ij}| \leq \sum_{(i,j)\in Q_K \setminus Q_{n_K}} |a_{ij}| \\ &= t_{KK} - t_{n_K n_K} < B - \left(B - \frac{\epsilon}{2}\right) = \frac{\epsilon}{2}. \end{aligned}$$

Now choose $N \geq 2N_1$ large enough that $|s_{KK} - S| < \epsilon/2$ for all $K \geq N$, which is possible since $(s_{KK}) \to S$. For $K \geq N$,

$$\begin{aligned} |D_K - S| &\leq |D_K - s_{KK}| + |s_{KK} - S| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

so $\sum_{k=2}^{\infty} d_k = \lim_{K\to\infty} D_K = S$.

Problem 2.8.7. Assume $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.
(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges so that we may apply Theorem 2.8.1.
(b) Let $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1$.

Solution. (a) The iterated sum of absolute values is $\alpha\beta$, where $\alpha = \sum_{i=1}^{\infty}|a_i|$ and $\beta = \sum_{j=1}^{\infty}|b_j|$, both finite by hypothesis.
Apply Theorem 2.8.1 to the array $a_{ij} = a_ib_j$. For fixed i the constant $|a_i|$ factors out of the partial sums, so by the Algebraic Limit Theorem for Series (Theorem 2.7.1),

$$\sum_{j=1}^{\infty}|a_ib_j| = |a_i|\sum_{j=1}^{\infty}|b_j| = |a_i|\beta,$$

which converges; this is the number b_i in the hypothesis of Theorem 2.8.1. Applying Theorem 2.7.1 once more,

$$\sum_{i=1}^{\infty}|a_i|\beta = \beta\sum_{i=1}^{\infty}|a_i| = \alpha\beta,$$

which converges as well. So $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges and Theorem 2.8.1 applies.
(b) s_{nn} factors.
Because the double sum is finite, it splits as a product of two partial sums:

$$s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right).$$

The two factors converge to A and B respectively, so the product rule of the Algebraic Limit Theorem (Theorem 2.3.3) gives

$$\lim_{n\to\infty} s_{nn} = AB.$$

By part (a) the hypothesis of Theorem 2.8.1 is met, and its conclusion is that both iterated sums equal $\lim_{n\to\infty} s_{nn}$:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = AB.$$

Finally, Exercise 2.8.6(b) applies under the same hypothesis and shows that the diagonal sum converges to $\lim_{n\to\infty} s_{nn}$ too. For this array the kth diagonal is

$$d_k = \sum_{i=1}^{k-1} a_ib_{k-i} = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1,$$

so $\sum_{k=2}^{\infty} d_k = AB$.

3 Basic Topology of R

Contents

3.1 Exercises 3.2.1–3.2.7

Problem 3.2.1. (a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?
(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbf{R}$.

Solution. (a) At the single line $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$. A finite set of positive reals has a *minimum*, and that minimum is again positive; an infinite set of positive reals need only have an infimum, possibly 0, and $\epsilon = 0$ produces no neighborhood at all. Everything before that line ($a \in \bigcap_k O_k$ yields $V_{\epsilon_k}(a) \subseteq O_k$ for each k) is valid for an arbitrary index set.
(b) Take $O_n = (-1/n,\, 1/n)$, each open by Example 3.2.2 (ii). Then

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, \tfrac{1}{n}\right) = \{0\},$$

since $|x| < 1/n$ for all $n \in \mathbf{N}$ forces $x = 0$ by the Archimedean Property (Theorem 1.4.2), while 0 lies in every O_n. The singleton $\{0\}$ has no limit points, hence is closed; it is nonempty and is not $\mathbf{R}$.

Problem 3.2.2. Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\} \quad \text{and} \quad B = \{x \in \mathbf{Q} : 0 < x < 1\}.$$

Answer the following questions for each set:
(a) What are the limit points?
(b) Is the set open? Closed?
(c) Does the set contain any isolated points?
(d) Find the closure of the set.

Solution. (a) The limit points of A are -1 and 1; the limit points of B are all of $[0, 1]$.
Write $a_n = (-1)^n + 2/n$. The odd terms $a_{2k-1} = -1 + \frac{2}{2k-1}$ decrease strictly to -1 and the even terms $a_{2k} = 1 + \frac{1}{k}$ decrease strictly to 1, so -1 and 1 are limit points by Theorem 3.2.5 (neither sequence takes its limit as a value). The odd terms lie in $[-1, 1]$ with $a_1 = 1$ and the even terms in $(1, 2]$, so all the a_n are distinct and any $x \notin \{-1, 1\}$ has a neighborhood containing only finitely many a_n; shrinking it further leaves at most x itself.
For B: if $y \in [0, 1]$ and $\epsilon > 0$, then $(y - \epsilon, y + \epsilon) \cap (0, 1)$ is a nonempty open interval, so by Theorem 1.4.3 it contains a rational $r \neq y$, and $r \in B$. No $y \notin [0, 1]$ qualifies, since $V_\epsilon(y) \cap (0, 1) = \emptyset$ for small ϵ.
(b) Neither set is open: every ϵ-neighborhood is uncountable, so it cannot sit inside the countable set A; and every ϵ-neighborhood of a point of B contains an irrational (Corollary 1.4.4), so it cannot sit inside B. Neither set is closed: -1 is a limit point of A with $-1 \notin A$ (all terms satisfy $a_n > -1$), and 0 is a limit point of B with $0 \notin B$.
(c) A: every point of A except $a_1 = 1$ is isolated. Indeed $1 = a_1 \in A$ is a limit point by (a), hence not isolated (Definition 3.2.6); for any other a_n, the distance to the nearest other term of the sequence is positive, and a neighborhood of that radius meets A only in a_n. B: no isolated points, since by (a) every point of B is a limit point of B.
(d) By Definition 3.2.11,

$$\overline{A} = A \cup \{-1, 1\} = A \cup \{-1\}, \qquad \overline{B} = B \cup [0, 1] = [0, 1].$$

Problem 3.2.3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed,

find a limit point that is not contained in the set.
(a) $\mathbf{Q}$.
(b) $\mathbf{N}$.
(c) $\{x \in \mathbf{R} : x \neq 0\}$.
(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.
(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$.

Solution. (a) Neither. Not open: no $V_\epsilon(0)$ lies in $\mathbf{Q}$, since every interval contains an irrational (Corollary 1.4.4). Not closed: $\sqrt{2}$ is a limit point of $\mathbf{Q}$ (Example 3.2.9 (iii): every real is) and $\sqrt{2} \notin \mathbf{Q}$ by Theorem 1.1.1.
(b) Closed, not open. Not open: $V_\epsilon(1)$ contains $1 + \epsilon/2 \notin \mathbf{N}$ for $\epsilon < 1$. Closed vacuously: $\mathbf{N}$ has no limit points, because $V_{1/2}(x)$ contains at most one natural number, hence meets $\mathbf{N}$ in no point other than possibly x itself.
(c) Open, not closed. Open: given $x \neq 0$, take $\epsilon = |x|$; then $V_\epsilon(x) \subseteq \mathbf{R} \setminus \{0\}$. (Equivalently, $\{0\}$ is closed and Theorem 3.2.13 applies.) Not closed: 0 is a limit point, since $V_\epsilon(0)$ contains $\epsilon/2 \neq 0$, and 0 is not in the set.
(d) Neither. Let $s_n = \sum_{k=1}^{n} 1/k^2$. These are exactly the partial sums of Example 2.4.4: (s_n) is increasing and bounded above (by 2), so $s_n \to L = \sup_n s_n$ by the Monotone Convergence Theorem (Theorem 2.4.2). Since $s_n < s_{n+1} \leq L$, no term equals L, so by Theorem 3.2.5 L is a limit point of the set that the set does not contain: not closed. Not open, since a neighborhood is uncountable and the set is countable.
(e) Closed, not open. Let $h_n = \sum_{k=1}^{n} 1/k$. The harmonic series diverges (Example 2.4.5), so $h_n \to \infty$; hence for any $x \in \mathbf{R}$ only finitely many h_n lie in $(x - 1, x + 1)$, and taking ϵ smaller than the distance from x to the nearest of those finitely many terms other than x gives $V_\epsilon(x) \cap \{h_n\} \subseteq \{x\}$. So the set has no limit points and is closed. Not open, for the countability reason in (d) (concretely, no $V_\epsilon(1)$ is contained in it).

Problem 3.2.4. Let A be nonempty and bounded above so that $s = \sup A$ exists.
(a) Show that $s \in \overline{A}$.
(b) Can an open set contain its supremum?

Solution. (a) If $s \in A$ we are done, since $\overline{A} = A \cup L \supseteq A$ (Definition 3.2.11). So assume $s \notin A$ and let $\epsilon > 0$. Because $s - \epsilon$ is not an upper bound of A (Lemma 1.3.8), there is $a \in A$ with

$$s - \epsilon < a \leq s,$$

and $a \neq s$ because $s \notin A$. Thus $V_\epsilon(s)$ meets A at a point other than s for every $\epsilon > 0$, so $s \in L$ by Definition 3.2.4, and again $s \in \overline{A}$.
(b) No. Suppose O is open and $s = \sup O$ exists with $s \in O$. By Definition 3.2.1 there is $\epsilon > 0$ with $V_\epsilon(s) \subseteq O$, and then $s + \epsilon/2 \in O$ exceeds the upper bound s, a contradiction.

Problem 3.2.5. Prove Theorem 3.2.8: a set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Solution. ($\Rightarrow$) Let F be closed and let $(x_n) \subseteq F$ be Cauchy. By the Cauchy Criterion (Theorem 2.6.4) $x_n \to x$ for some $x \in \mathbf{R}$. If $x_n = x$ for some n, then $x \in F$ immediately. Otherwise $x_n \neq x$ for all n, so x is a limit point of F by Theorem 3.2.5, and $x \in F$ because F contains its limit points (Definition 3.2.7).
($\Leftarrow$) Assume every Cauchy sequence in F has its limit in F, and let x be a limit point of F. By Theorem 3.2.5 there is a sequence $(x_n) \subseteq F$ with $x_n \neq x$ and $x_n \to x$. Convergent sequences are Cauchy (Theorem 2.6.2), so (x_n) is a Cauchy sequence contained in F and therefore $x = \lim x_n \in F$. Thus F contains its limit points and is closed.

Problem 3.2.6. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.
(a) An open set that contains every rational number must necessarily be all of $\mathbf{R}$.
(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."
(c) Every nonempty open set contains a rational number.
(d) Every bounded infinite closed set contains a rational number.
(e) The Cantor set is closed.

Solution. (a) False. $O = \mathbf{R} \setminus \{\sqrt{2}\}$ contains $\mathbf{Q}$ (Theorem 1.1.1), is open by the argument of Exercise 3.2.3 (c) applied at $\sqrt{2}$, and is not $\mathbf{R}$.
(b) False. Put $F_n = [n, \infty)$. Each F_n is closed (a limit of a sequence in F_n satisfies $x \geq n$ by the Order Limit Theorem, Theorem 2.3.4), and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$, but

$$\bigcap_{n=1}^{\infty} F_n = \emptyset,$$

since any $x \in \mathbf{R}$ fails $x \geq n$ for n large (Archimedean Property, Theorem 1.4.2).
(c) True. Let $O \neq \emptyset$ be open and pick $a \in O$. By Definition 3.2.1 there is $\epsilon > 0$ with $V_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq O$, and Theorem 1.4.3 supplies a rational r with $a - \epsilon < r < a + \epsilon$. Then $r \in O$.
(d) False. Take

$$A = \left\{ \sqrt{2}\left(1 + \tfrac{1}{n}\right) : n \in \mathbf{N} \right\} \cup \{\sqrt{2}\}.$$

Every element is irrational: $\sqrt{2}\,q$ with $q \in \mathbf{Q}$, $q > 0$, cannot be rational, or else $\sqrt{2}$ would be. A is infinite, and $A \subseteq [\sqrt{2}, 2\sqrt{2}]$ is bounded. It is closed: the only limit point is $\sqrt{2}$ (the terms decrease strictly to $\sqrt{2}$, so each is isolated), and $\sqrt{2} \in A$.
(e) True. The Cantor set is $C = \bigcap_{n=0}^{\infty} C_n$, where each C_n is a union of 2^n closed intervals. A finite union of closed sets is closed (Theorem 3.2.14 (i)), so each C_n is closed; an arbitrary intersection of closed sets is closed (Theorem 3.2.14 (ii)), so C is closed.

Problem 3.2.7. Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.
(a) Show that the set L is closed.
(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Solution. ($\star$) If $y \in L$ and $0 < |x - y| < \epsilon$, then $V_\epsilon(x)$ contains a point of A other than x. Indeed, set

$$\delta = \min\{\, |x-y|,\ \epsilon - |x-y| \,\} > 0.$$

Because $y \in L$, Definition 3.2.4 gives a point $a \in A \cap V_\delta(y)$ with $a \neq y$. Then

$$\begin{aligned} &|a - x| \leq |a - y| + |y - x| < \delta + |x - y| \leq \epsilon, \\ &|a - y| < \delta \leq |x - y| \implies a \neq x. \end{aligned}$$

Hence $a \in A \cap V_\epsilon(x)$ with $a \neq x$, proving ($\star$).
(a) Let x be a limit point of L and let $\epsilon > 0$. Then $V_\epsilon(x)$ contains some $y \in L$ with $y \neq x$, and ($\star$) produces a point of A in $V_\epsilon(x)$ different from x. As ϵ was arbitrary, $x \in L$, so L contains its limit points and is closed (Definition 3.2.7).
(b) Let x be a limit point of $A \cup L$ and let $\epsilon > 0$. Choose $z \in (A \cup L) \cap V_\epsilon(x)$ with $z \neq x$. Two cases:
(i) $z \in A$: then $V_\epsilon(x)$ already meets A at a point other than x.
(ii) $z \in L$: apply ($\star$) with $y = z$.
Either way $V_\epsilon(x)$ meets A away from x, so $x \in L$.
Theorem 3.2.12 now follows. *Closed:* by the previous paragraph every limit point of $\overline{A} = A \cup L$ lies in $L \subseteq \overline{A}$, so $\overline{A}$ is closed. *Smallest:* let F be any closed set with $A \subseteq F$ and let $x \in L$. Every $V_\epsilon(x)$ meets

A at a point other than x, hence meets F at a point other than x, so x is a limit point of F and thus $x \in F$. Therefore

$$\overline{A} = A \cup L \subseteq F,$$

and $\overline{A}$ is the smallest closed set containing A.

3.2 Exercises 3.2.8–3.2.14

Problem 3.2.8. Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.
(a) $\overline{A \cup B}$
(b) $A \setminus B = \{x \in A : x \notin B\}$
(c) $(A^c \cup B)^c$
(d) $(A \cap B) \cup (A^c \cap B)$
(e) $\overline{A}^c \cap \overline{A^c}$

Solution. (a) Definitely closed, not definitely open. Every closure is closed (Theorem 3.2.12). With $A = (0, 1)$ and $B = \{2\}$ we get $\overline{A \cup B} = [0, 1] \cup \{2\}$, and no $V_\epsilon(0)$ lies inside this set.
(b) Definitely open, not definitely closed. Indeed $A \setminus B = A \cap B^c$, where B^c is open by Theorem 3.2.13, so the intersection of two open sets is open by Theorem 3.2.3 (ii). With $A = (0, 1)$ and $B = \{1/2\}$ we get $(0, 1/2) \cup (1/2, 1)$, which omits its limit point $1/2$.
(c) Definitely open, not definitely closed: this is the same set as (b), since

$$(A^c \cup B)^c = (A^c)^c \cap B^c = A \cap B^c = A \setminus B.$$

(d) Definitely closed, not definitely open, because the set is exactly B:

$$(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = \mathbf{R} \cap B = B.$$

Taking $B = [0, 1]$ shows it need not be open.
(e) Definitely open, not definitely closed, because the set is exactly $\overline{A}^c$. From $A \subseteq \overline{A}$ and $A^c \subseteq \overline{A^c}$ we get

$$\overline{A}^c \subseteq A^c \subseteq \overline{A^c}, \qquad \text{hence} \qquad \overline{A}^c \cap \overline{A^c} = \overline{A}^c,$$

which is open by Theorems 3.2.12 and 3.2.13. Taking $A = (0, 1)$ gives $\overline{A}^c = (-\infty, 0) \cup (1, \infty)$, which omits its limit point 0.

Problem 3.2.9. (De Morgan's Laws). A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.
(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\Big(\bigcup_{\lambda \in \Lambda} E_\lambda\Big)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c \qquad \text{and} \qquad \Big(\bigcap_{\lambda \in \Lambda} E_\lambda\Big)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c.$$

(b) Now, provide the details for the proof of Theorem 3.2.14.

Solution. (a) Both identities are a chain of equivalences on membership. For the first,

$$\begin{aligned} x \in \Big(\bigcup_{\lambda} E_\lambda\Big)^c &\iff \neg\big(\exists\, \lambda \in \Lambda \text{ with } x \in E_\lambda\big) \\ &\iff \forall\, \lambda \in \Lambda,\ x \notin E_\lambda \\ &\iff \forall\, \lambda \in \Lambda,\ x \in E_\lambda^c \iff x \in \bigcap_{\lambda} E_\lambda^c, \end{aligned}$$

and for the second, replacing the existential quantifier by a universal one,

$$x \in \Big(\bigcap_{\lambda} E_\lambda\Big)^c \iff \neg(\forall\, \lambda \in \Lambda,\ x \in E_\lambda)$$

$$\iff \exists\, \lambda \in \Lambda \text{ with } x \in E_\lambda^c \iff x \in \bigcup_{\lambda} E_\lambda^c.$$

(b) Theorem 3.2.14 asserts that (i) the union of a finite collection of closed sets is closed and (ii) the intersection of an arbitrary collection of closed sets is closed. Let $\{F_\lambda : \lambda \in \Lambda\}$ be closed sets, so each F_λ^c is open by Theorem 3.2.13.
For (ii), part (a) gives

$$\Big(\bigcap_{\lambda\in\Lambda} F_\lambda\Big)^c = \bigcup_{\lambda\in\Lambda} F_\lambda^c,$$

an arbitrary union of open sets, hence open by Theorem 3.2.3 (i); so $\bigcap_\lambda F_\lambda$ is closed, again by Theorem 3.2.13.
For (i), with $F_1, \ldots, F_N$ closed, part (a) gives

$$\Big(\bigcup_{k=1}^{N} F_k\Big)^c = \bigcap_{k=1}^{N} F_k^c,$$

a finite intersection of open sets – finiteness being precisely the hypothesis Theorem 3.2.3 (ii) demands – hence open; so $\bigcup_{k=1}^N F_k$ is closed.

Problem 3.2.10. Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.
(i) A countable set contained in $[0,1]$ with no limit points.
(ii) A countable set contained in $[0,1]$ with no isolated points.
(iii) A set with an uncountable number of isolated points.

Solution. Only (ii) is realizable, and $A = \mathbf{Q} \cap [0,1]$ does it. It is an infinite subset of the countable set $\mathbf{Q}$ (Theorem 1.5.6 (i)), hence countable by Theorem 1.5.7, and no $q \in A$ is isolated: given $\epsilon > 0$, the set $V_\epsilon(q) \cap [0,1]$ contains a nondegenerate interval with endpoint q (because $0 \le q \le 1$), so by the density of $\mathbf{Q}$ (Theorem 1.4.3) it contains a rational $r \ne q$, whence $V_\epsilon(q)$ meets A in a point other than q and q is a limit point of A (Definition 3.2.4, Definition 3.2.6).
(i) is impossible. By Definition 1.5.5 a countable set is infinite, so write $A = \{a_1, a_2, a_3, \ldots\}$ with the a_n distinct. The sequence (a_n) lies in $[0,1]$ and is therefore bounded, so the Bolzano–Weierstrass Theorem (Theorem 2.5.5) supplies a convergent subsequence $a_{n_k} \to x$. The terms are distinct, so at most one a_{n_k} equals x; deleting that one term leaves a sequence in A with $a_{n_k} \ne x$ still converging to x, and Theorem 3.2.5 makes x a limit point of A.
(iii) is impossible: the isolated points of any set $A \subseteq \mathbf{R}$ inject into $\mathbf{Q} \times \mathbf{Q}$. If x is isolated, Definitions 3.2.4 and 3.2.6 give an $\epsilon > 0$ with $V_\epsilon(x) \cap A = \{x\}$, and Theorem 1.4.3 provides rationals

$$p_x \in (x-\epsilon, x), \qquad q_x \in (x, x+\epsilon), \qquad \text{so} \qquad (p_x, q_x) \cap A = \{x\}.$$

The map $x \mapsto (p_x, q_x)$ is one-to-one on the isolated points: if $x \ne y$ were isolated with $(p_x, q_x) = (p_y, q_y)$, then $y \in (p_y, q_y) \cap A = (p_x, q_x) \cap A = \{x\}$, forcing $y = x$. Now $\mathbf{Q} \times \mathbf{Q} = \bigcup_{q\in\mathbf{Q}} (\{q\} \times \mathbf{Q})$ is a countable union of countable sets (re-indexed by $\mathbf{N}$ using Theorem 1.5.6 (i)), hence countable by Theorem 1.5.8 (ii). Its subset $\{(p_x, q_x) : x \text{ isolated}\}$ is therefore countable or finite by Theorem 1.5.7, and the injection puts the isolated points of A in 1–1 correspondence with it.

Problem 3.2.11. (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) Does this result about closures extend to infinite unions of sets?

Solution. (a) Both inclusions are minimality statements read off Theorem 3.2.12, which says $\overline{E}$ is the smallest closed set containing E.
($\subseteq$) The set $\overline{A} \cup \overline{B}$ is a union of two closed sets, hence closed by Theorem 3.2.14 (i), and it contains $A \cup B$. Minimality of $\overline{A \cup B}$ gives $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.
($\supseteq$) The set $\overline{A \cup B}$ is closed and contains A, so minimality of $\overline{A}$ gives $\overline{A} \subseteq \overline{A \cup B}$; symmetrically $\overline{B} \subseteq \overline{A \cup B}$. Hence

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

(b) No. Only the inclusion $\bigcup_n \overline{A_n} \subseteq \overline{\bigcup_n A_n}$ survives, by the ($\supseteq$) argument above, which used nothing about the number of sets. For the failure, enumerate $\mathbf{Q} \cap [0,1] = \{r_1, r_2, r_3, \ldots\}$ (an infinite subset of $\mathbf{Q}$, so countable by Theorems 1.5.6 (i) and 1.5.7) and set $A_n = \{r_n\}$. A singleton has no limit points, hence is closed by Definition 3.2.7, so $\overline{A_n} = A_n$, while $\bigcup_n A_n = \mathbf{Q} \cap [0,1]$ has closure $[0,1]$ by the density of $\mathbf{Q}$ (Theorem 1.4.3):

$$\bigcup_{n=1}^{\infty} \overline{A_n} = \mathbf{Q} \cap [0,1] \neq [0,1] = \overline{\bigcup_{n=1}^{\infty} A_n}\,.$$

Problem 3.2.12. Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both

$$\{x : x \in A \text{ and } x < s\} \qquad \text{and} \qquad \{x : x \in A \text{ and } x > s\}$$

are uncountable. Show B is nonempty and open.

Solution. Write $B = U \cap V$ where

$$U = \{s \in \mathbf{R} : A \cap (-\infty, s) \text{ is uncountable}\}, \quad V = \{s \in \mathbf{R} : A \cap (s, \infty) \text{ is uncountable}\},$$

and call a set *small* if it is finite or countable, so that uncountable means not small (Definition 1.5.5). Two facts are used throughout. A subset of a small set is small (Theorem 1.5.7). And if $S_1, S_2, S_3, \ldots$ are small, so is $\bigcup_n S_n$: each $S_n \cup \mathbf{N}$ is countable (by Theorem 1.5.8 (i) when S_n is countable; by listing the finitely many points of $S_n \setminus \mathbf{N}$ ahead of $\mathbf{N}$ when S_n is finite), so $\bigcup_n (S_n \cup \mathbf{N})$ is countable by Theorem 1.5.8 (ii), and Theorem 1.5.7 passes this to the subset $\bigcup_n S_n$.
Openness. Suppose $s \in U$. Any $x < s$ satisfies $1/n < s - x$ for some n (Theorem 1.4.2 (ii)), so

$$A \cap (-\infty, s) = \bigcup_{n=1}^{\infty} A \cap (-\infty, s - 1/n),$$

and the left side is not small, so some term $A \cap (-\infty, s - 1/n_0)$ is not small, i.e. $s - 1/n_0 \in U$. But U is closed upward: if $t > u \in U$ then $A \cap (-\infty, u) \subseteq A \cap (-\infty, t)$, and a superset of an uncountable set is uncountable, so $t \in U$. Hence

$$V_{1/n_0}(s) \subseteq (s - 1/n_0, \infty) \subseteq U,$$

and U is open by Definition 3.2.1. Symmetrically V is open and closed downward. Consequently $B = U \cap V$ is open by Theorem 3.2.3 (ii).
Nonemptiness. First, $U \neq \emptyset$: by the Archimedean Property (Theorem 1.4.2) every real lies below some $n \in \mathbf{N}$, so $A = \bigcup_{n=1}^{\infty} A \cap (-\infty, n)$, and were every term small A would be small. Some n therefore has $n \in U$, and symmetrically $V \neq \emptyset$.
Now suppose, for contradiction, that $U \cap V = \emptyset$. Every $v \in V$ satisfies $v < u$ for every $u \in U$, since $v = u$ or $v > u$ would each place v in U (upward closure), hence in $U \cap V$. Thus each $v \in V$ is a lower bound for U and each $u \in U$ an upper bound for V; both sets being nonempty, $c = \inf U$ and $\sup V$ exist (Axiom of Completeness and Exercise 1.3.3), and $\sup V \leq \inf U = c$. For each $n \in \mathbf{N}$,

$$\begin{aligned} c - 1/n < \inf U \implies c - 1/n \notin U \quad &\implies A \cap (-\infty, c - 1/n) \text{ is small},\\ c + 1/n > \sup V \implies c + 1/n \notin V \quad &\implies A \cap (c + 1/n, \infty) \text{ is small}. \end{aligned}$$

Taking countable unions over n makes $A \cap (-\infty, c)$ and $A \cap (c, \infty)$ small, whence

$$A \subseteq (A \cap (-\infty, c)) \cup \{c\} \cup (A \cap (c, \infty))$$

is small – contradicting the hypothesis that A is uncountable. Hence $B = U \cap V \neq \emptyset$.

Problem 3.2.13. Prove that the only sets that are both open and closed are $\mathbf{R}$ and the empty set $\emptyset$.

Solution. Both $\mathbf{R}$ and $\emptyset$ do qualify: each is open by Example 3.2.2 (i), and each is the complement of the other, so each is closed by Theorem 3.2.13. For the converse, suppose E is both open and closed with $E \neq \emptyset$ and $E \neq \mathbf{R}$, and pick $a \in E$ and $b \in E^c$; assume $a < b$, the case $b < a$ being the mirror image with $\inf\bigl(E \cap [b, a]\bigr)$ in place of the supremum below.
Let

$$S = E \cap [a, b], \qquad s = \sup S,$$

which exists because S is nonempty ($a \in S$) and bounded above by b. By Exercise 3.2.4 (a), $s \in \overline{S}$, and $\overline{S} \subseteq \overline{E} = E$ since $\overline{E}$ is a closed set containing S (Theorem 3.2.12) and E is closed (Exercise 3.2.14 (a)). So $s \in E$, while $b \notin E$; since also $a \in S$, this gives $a \leq s < b$.
Now use that E is open: there is $\epsilon > 0$ with $V_\epsilon(s) \subseteq E$, and shrinking ϵ we may assume $\epsilon < b - s$. Then $a \leq s < s + \epsilon/2 < s + \epsilon < b$, so

$$s + \epsilon/2 \in V_\epsilon(s) \cap [a, b] \subseteq E \cap [a, b] = S,$$

while $s + \epsilon/2 > s = \sup S$ – a contradiction.

Problem 3.2.14. A dual notion to the closure of a set is the interior of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \text{there exists } V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.
(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.
(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Solution. (a) Let L be the set of limit points of E, so $\overline{E} = E \cup L$ by Definition 3.2.11. Then

$$\overline{E} = E \iff L \subseteq E \iff E \text{ contains its limit points} \iff E \text{ is closed},$$

the last step being Definition 3.2.7.
For the interior, $E^\circ \subseteq E$ holds by definition, so the content is the reverse inclusion. If E is open, then every $x \in E$ admits $V_\epsilon(x) \subseteq E$ (Definition 3.2.1), which is exactly the requirement for $x \in E^\circ$; hence $E \subseteq E^\circ$ and $E^\circ = E$. Conversely, if $E^\circ = E$ then every $x \in E$ has some $V_\epsilon(x) \subseteq E$, which is Definition 3.2.1 verbatim.
(b) The first identity is a membership chain. For $x \in \mathbf{R}$,

$$\begin{aligned} x \in \overline{E}^c &\iff x \notin E \ \wedge \ x \text{ is not a limit point of } E \\ &\iff x \notin E \ \wedge \ \exists \epsilon > 0 : V_\epsilon(x) \cap E \subseteq \{x\} \\ &\iff \exists \epsilon > 0 : V_\epsilon(x) \cap E = \emptyset \\ &\iff \exists \epsilon > 0 : V_\epsilon(x) \subseteq E^c \iff x \in (E^c)^\circ. \end{aligned}$$

The third equivalence is where $x \notin E$ is spent: it upgrades $V_\epsilon(x) \cap E \subseteq \{x\}$ to $V_\epsilon(x) \cap E = \emptyset$, and conversely $V_\epsilon(x) \cap E = \emptyset$ forces $x \notin E$ since $x \in V_\epsilon(x)$.
The second identity now follows by applying the first to the set E^c:

$$\overline{E^c}^{\,c} = \bigl((E^c)^c\bigr)^\circ = E^\circ, \qquad \text{so} \qquad \overline{E^c} = (E^\circ)^c.$$

3.3 Exercises 3.2.15–3.3.6

Problem 3.2.15. A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.
(a) Show that a closed interval $[a, b]$ is a G_δ set.
(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.
(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set. (We will see in Section 3.5 that $\mathbf{Q}$ is not a G_δ set, nor is $\mathbf{I}$ an F_σ set.)

Solution. (a) Fatten symmetrically:

$$[a, b] \;=\; \bigcap_{n=1}^{\infty}\Big(a - \tfrac{1}{n},\; b + \tfrac{1}{n}\Big),$$

a countable intersection of open sets. The inclusion $\subseteq$ is immediate; conversely, if $a - 1/n < x < b + 1/n$ for every n, then $x > b$ would force $0 < x - b < 1/n$ for all $n \in \mathbf{N}$, contradicting the Archimedean Property (Theorem 1.4.2), and symmetrically $x < a$ is impossible, so $x \in [a, b]$.
(b) Fatten only on the right, and exhaust from the left (here $a < b$):

$$(a, b] = \bigcap_{n=1}^{\infty}\Big(a,\; b + \tfrac{1}{n}\Big)$$

$$(a, b] = \bigcup_{n=1}^{\infty}\Big[a + \tfrac{b-a}{n},\; b\Big].$$

In the first, each set is open and a point of the intersection has $x > a$, while $x > b$ is excluded exactly as in (a), so $a < x \le b$. In the second, each $[a + (b - a)/n, b]$ is closed (Example 3.2.9 (ii)) and sits inside $(a, b]$; conversely $a < x \le b$ lands in the nth set once the Archimedean Property (Theorem 1.4.2) supplies n with $(b - a)/n < x - a$.
(c) Enumerate $\mathbf{Q} = \{q_1, q_2, q_3, \dots\}$, possible since $\mathbf{Q}$ is countable (Theorem 1.5.6 (i)). A singleton has no limit points at all (for $x \ne q_n$ take $\epsilon = |x - q_n|$; for $x = q_n$ every $V_\epsilon(x)$ meets $\{q_n\}$ only in x), so it is vacuously closed (Definition 3.2.7) and

$$\mathbf{Q} \;=\; \bigcup_{n=1}^{\infty}\{q_n\}$$

is an F_σ set. Complementing and applying De Morgan's Laws (Exercise 3.2.9),

$$\mathbf{I} \;=\; \mathbf{Q}^c \;=\; \Big(\bigcup_{n=1}^{\infty}\{q_n\}\Big)^c \;=\; \bigcap_{n=1}^{\infty}\{q_n\}^c,$$

where each $\{q_n\}^c$ is open by Theorem 3.2.13; thus $\mathbf{I}$ is a G_δ set.

Problem 3.3.1. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Solution. Both are attained, because each is the limit of a sequence drawn from K and K is closed. By Theorem 3.3.4 a compact K is closed and bounded. Since K is nonempty and bounded above, the Axiom of Completeness supplies $s = \sup K$. For each $n \in \mathbf{N}$ the number $s - 1/n$ is not an upper bound for K, so we may pick $x_n \in K$ with

$$s - \frac{1}{n} < x_n \le s.$$

Squeezing gives $(x_n) \to s$, so (x_n) is a Cauchy sequence contained in K, and Theorem 3.2.8 (the closedness criterion) forces $s \in K$.

Symmetrically, K is bounded below, so $i = \inf K$ exists; choosing $y_n \in K$ with $i \leq y_n < i + 1/n$ gives $(y_n) \to i$ and hence $i \in K$ by the same theorem.

Problem 3.3.2. Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.
(a) $\mathbf{N}$.
(b) $\mathbf{Q} \cap [0, 1]$.
(c) The Cantor set.
(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.
(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

Solution. Compact: (c) and (e). Not compact: (a), (b), (d).
(a) Take $x_n = n$. Any subsequence satisfies $x_{n_k} = n_k \geq k$, so it is unbounded and therefore divergent (Theorem 2.3.2: convergent sequences are bounded). Definition 3.3.1 fails at the very first step.
(b) Take rationals $r_n \in [0, 1]$ with $(r_n) \to 1/\sqrt{2}$, available by the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 3.2.10). Every subsequence converges to the same limit $1/\sqrt{2}$ (Theorem 2.5.2), and $1/\sqrt{2} \notin \mathbf{Q}$. The set is bounded but not closed.
(c) Compact. With $C = \bigcap_{n=0}^{\infty} C_n$ as in Section 3.1, each C_n is a union of 2^n closed intervals, hence closed by Theorem 3.2.14 (i), and an arbitrary intersection of closed sets is closed by Theorem 3.2.14 (ii). Since $C \subseteq [0, 1]$ is bounded, Theorem 3.3.4 gives compactness.
(d) Write $s_n = \sum_{k=1}^{n} 1/k^2$, so the set is $S = \{s_n : n \in \mathbf{N}\}$. The sequence (s_n) is strictly increasing and bounded, hence convergent by Theorem 2.4.2 (this is exactly Example 2.4.4), with

$$L = \lim s_n = \sup S.$$

Strict monotonicity gives $s_n < s_{n+1} \leq L$ for every n, so $L \notin S$. By Theorem 2.5.2 every subsequence of (s_n) converges to L, so this sequence in S has no subsequence with limit in S: S is bounded but not closed.
(e) Compact. The set is $E = \{1\} \cup \{(n-1)/n : n \geq 2\}$, so $E \subseteq [0, 1]$ is bounded. A sequence of distinct points of E must use indices $n \to \infty$, since only finitely many elements of E lie below any given $(N-1)/N$; such a sequence converges to 1. So by Theorem 3.2.5 the only limit point of E is $1 \in E$, making E closed (Definition 3.2.7), and Theorem 3.3.4 gives compactness.

Problem 3.3.3. Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbf{R}$ is closed and bounded, then it is compact.

Solution. Bolzano–Weierstrass produces the subsequence and closedness catches its limit; together with the argument in the text this completes the proof of Theorem 3.3.4.
Let (a_n) be an arbitrary sequence contained in K. Boundedness of K (Definition 3.3.3) gives an $M > 0$ with $|a_n| \leq M$ for all n, so (a_n) is a bounded sequence and Theorem 2.5.5 (Bolzano–Weierstrass) supplies a convergent subsequence

$$(a_{n_k}) \to a.$$

A convergent sequence is Cauchy, and (a_{n_k}) is a Cauchy sequence contained in the closed set K, so $a \in K$ by Theorem 3.2.8. Thus every sequence in K has a subsequence converging to a limit in K, which is Definition 3.3.1.

Problem 3.3.4. Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.
(a) $K \cap F$
(b) $F^c \cup K^c$
(c) $K \backslash F = \{x \in K : x \notin F\}$

(d) $K \cap F^c$

Solution. (a) both; (b), (c), (d) neither.
(a) K is closed and bounded by Theorem 3.3.4, so $K \cap F$ is closed by Theorem 3.2.14 (ii) and is bounded because $K \cap F \subseteq K$; being closed and bounded it is compact, again by Theorem 3.3.4.
(b) De Morgan gives $F^c \cup K^c = (F \cap K)^c$, which is open by Theorem 3.2.13. It is never compact: with $M > 0$ chosen so that $K \subseteq [-M, M]$,

$$F^c \cup K^c \supseteq K^c \supseteq (M, \infty),$$

so the set is unbounded. Nor need it be closed: $K = F = [0, 1]$ gives $(-\infty, 0) \cup (1, \infty)$, which omits its limit point 0.
(c) The set $K \backslash F \subseteq K$ is bounded, but $K = [0, 1]$ and $F = \{0\}$ give $K \backslash F = (0, 1]$, which omits its limit point 0 and so is not closed (Definition 3.2.7), hence not compact (Theorem 3.3.4).
(d) $K \cap F^c$ is literally the set $K \backslash F$ of part (c), so the same example settles it: neither.

Problem 3.3.5. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.
(a) The arbitrary intersection of compact sets is compact.
(b) The arbitrary union of compact sets is compact.
(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.
(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution. (a) True; (b), (c), (d) false.
(a) Let $\{K_\lambda : \lambda \in \Lambda\}$ be a nonempty collection of compact sets, so each K_λ is closed and bounded (Theorem 3.3.4). The intersection is closed by Theorem 3.2.14 (ii), and fixing any $\lambda_0 \in \Lambda$ gives

$$\bigcap_{\lambda \in \Lambda} K_\lambda \subseteq K_{\lambda_0},$$

so it is bounded. Theorem 3.3.4 returns compactness.
(b) False. Each singleton $\{n\}$ is compact (it is closed and bounded), but

$$\bigcup_{n \in \mathbf{N}} \{n\} = \mathbf{N}$$

is unbounded, hence not compact.
(c) False. Take $A = (0, 1)$ and $K = [0, 1]$. Then $A \cap K = (0, 1)$, and the sequence $(1/n)_{n \geq 2}$ lies in it while every subsequence converges to $0 \notin (0, 1)$ (Theorem 2.5.2), contradicting Definition 3.3.1.
(d) False. Put $F_n = [n, \infty)$. Each is nonempty and closed (its complement $(-\infty, n)$ is open, Theorem 3.2.13) and the sequence is nested, yet the Archimedean Property gives

$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset,$$

since no real number exceeds every natural number.

Problem 3.3.6. This exercise is meant to illustrate the point made in the opening paragraph to Section 3.3. Verify that the following three statements are true if every blank is filled in with the word "finite." Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the word "closed"?
(a) Every ____ set has a maximum.

(b) If A and B are ____, then $A+B=\{a+b : a\in A, b\in B\}$ is also ____.
(c) If $\{A_n : n\in\mathbf{N}\}$ is a collection of ____ sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Solution. All three hold for “finite” and all three hold for “compact”; all three fail for “closed.” (Read the sets in (a) as nonempty: the empty set is finite and compact but has no maximum.)
(a) Finite: induct on $|A|$; a set with one element is its own maximum, and $\max(A\cup\{x\}) = \max\{\max A, x\}$. Compact: Exercise 3.3.1 shows $\sup K\in K$, and an element of K that is an upper bound for K is exactly $\max K$. Closed: false — $\mathbf{N}$ is closed (it has no limit points, so Definition 3.2.7 holds vacuously) and has no maximum.
(b) Finite: $A+B$ is the image of $A\times B$ under addition, so $|A+B|\le |A|\cdot|B|<\infty$.
Compact: let (a_n+b_n) be a sequence in $A+B$. Definition 3.3.1 applied to A gives a subsequence $(a_{n_k})\to a\in A$; applied to (b_{n_k}) in B it gives a further subsequence $(b_{n_{k_j}})\to b\in B$. Since $(a_{n_{k_j}})$ still tends to a (Theorem 2.5.2), the Algebraic Limit Theorem (Theorem 2.3.3) yields

$$a_{n_{k_j}}+b_{n_{k_j}} \longrightarrow a+b\in A+B.$$

Closed: false. Take

$$A=\mathbf{N},\qquad B=\Big\{-n+\tfrac{1}{n} : n\ge 2\Big\}.$$

Consecutive points of B satisfy $b_n-b_{n+1}=1+\frac{1}{n}-\frac{1}{n+1}>1$, so B has no limit points and is closed, as is $\mathbf{N}$. Now $n+(-n+1/n)=1/n\in A+B$ for every $n\ge 2$, so 0 is a limit point of $A+B$; but $m-n+1/n=0$ would force $1/n=n-m\in\mathbf{Z}$, impossible for $n\ge 2$. Hence $0\notin A+B$ and $A+B$ is not closed.
(c) Set $B_N=\bigcap_{n=1}^{N} A_n$, so (B_N) is nested and each $B_N\ne\emptyset$ by hypothesis, and $\bigcap_{n=1}^{\infty} A_n=\bigcap_{N=1}^{\infty} B_N$.
Finite: each B_N is finite, so the integers $|B_N|$ are non-increasing and at least 1, hence constant for $N\ge N_0$; with $B_N\subseteq B_{N_0}$ this forces $B_N=B_{N_0}$ for all $N\ge N_0$, and the intersection equals $B_{N_0}\ne\emptyset$.
Compact: each B_N is compact by part (a) of Exercise 3.3.5, so Theorem 3.3.5 (Nested Compact Set Property) applies to the nested nonempty compact sets $B_1\supseteq B_2\supseteq\cdots$ and gives $\bigcap_N B_N\ne\emptyset$.
Closed: false — with $A_n=[n,\infty)$ every finite subcollection meets in $[\max_i n_i,\infty)\ne\emptyset$, while $\bigcap_{n=1}^{\infty} A_n=\emptyset$.

3.4 Exercises 3.3.7–3.3.13

Problem 3.3.7. As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C+C=\{x+y : x,y\in C\}$ is equal to the closed interval $[0,2]$. (Keep in mind that C has zero length and contains no intervals.)
Because $C\subseteq[0,1]$, $C+C\subseteq[0,2]$, so we only need to prove the reverse inclusion $[0,2]\subseteq\{x+y : x,y\in C\}$. Thus, given $s\in[0,2]$, we must find two elements $x,y\in C$ satisfying $x+y=s$.
(a) Show that there exist $x_1,y_1\in C_1$ for which $x_1+y_1=s$. Show in general that, for an arbitrary $n\in\mathbf{N}$, we can always find $x_n,y_n\in C_n$ for which $x_n+y_n=s$.
(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x+y=s$.

Solution. (a) $C_n+C_n=[0,2]$ for every $n\ge 0$, so the required x_n,y_n always exist.
For $n=1$, use $[a,b]+[c,d]=[a+c,b+d]$ on each of the three pairings of the two components of $C_1=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$:

$$[0,\tfrac{1}{3}]+[0,\tfrac{1}{3}]=[0,\tfrac{2}{3}],$$
$$[0,\tfrac{1}{3}]+[\tfrac{2}{3},1]=[\tfrac{2}{3},\tfrac{4}{3}],$$
$$[\tfrac{2}{3},1]+[\tfrac{2}{3},1]=[\tfrac{4}{3},2],$$

whose union is $[0,2]$. In particular $s\in[0,2]$ lies in one of these three sets, which hands us $x_1,y_1\in C_1$ with $x_1+y_1=s$.
For general n, induct, using the self-similarity built into the construction of Section 3.1 — passing from

C_n to C_{n+1} removes middle thirds inside $[0, \frac{1}{3}]$ and inside $[\frac{2}{3}, 1]$ in exactly the scaled-by-$\frac{1}{3}$ pattern, so

$$C_{n+1} = \tfrac{1}{3}C_n \ \cup \ \Big(\tfrac{2}{3} + \tfrac{1}{3}C_n\Big).$$

The base case is $C_0 + C_0 = [0,1] + [0,1] = [0,2]$. Assuming $C_n + C_n = [0,2]$, the three pairings of the two pieces of C_{n+1} give

$$\begin{aligned} \tfrac{1}{3}C_n + \tfrac{1}{3}C_n &= \tfrac{1}{3}[0,2] = [0, \tfrac{2}{3}], \\ \tfrac{1}{3}C_n + (\tfrac{2}{3} + \tfrac{1}{3}C_n) &= \tfrac{2}{3} + \tfrac{1}{3}[0,2] = [\tfrac{2}{3}, \tfrac{4}{3}], \\ (\tfrac{2}{3} + \tfrac{1}{3}C_n) + (\tfrac{2}{3} + \tfrac{1}{3}C_n) &= \tfrac{4}{3} + \tfrac{1}{3}[0,2] = [\tfrac{4}{3}, 2], \end{aligned}$$

and their union is $C_{n+1} + C_{n+1} = [0,2]$.
(b) Extract a convergent subsequence from (x_n) and let (y_n) follow it.
Fix $s \in [0,2]$ and choose $x_n, y_n \in C_n$ with $x_n + y_n = s$ as in (a). The sequence (x_n) lies in $[0,1]$, so Theorem 2.5.5 (Bolzano–Weierstrass) supplies a convergent subsequence $(x_{n_k}) \to x$, and then by the Algebraic Limit Theorem (Theorem 2.3.3)

$$y_{n_k} = s - x_{n_k} \longrightarrow s - x =: y,$$

so $x + y = s$.
It remains to see that $x, y \in C$. Fix $m \in \mathbf{N}$. The sets C_n are nested downward, so for every k with $n_k \geq m$ we have $x_{n_k} \in C_{n_k} \subseteq C_m$. That tail of (x_{n_k}) is a convergent, hence Cauchy, sequence contained in the closed set C_m (a finite union of closed intervals, closed by Theorem 3.2.14 (i)), so $x \in C_m$ by Theorem 3.2.8. The same argument applies verbatim to $y_{n_k} \in C_{n_k} \subseteq C_m$, giving $y \in C_m$. Since m was arbitrary,

$$x, y \in \bigcap_{m=0}^{\infty} C_m = C,$$

and $x+y = s$. Hence $[0,2] \subseteq C+C$, and with the reverse inclusion noted in the statement, $C+C = [0,2]$.

Problem 3.3.8. Let K and L be nonempty compact sets, and define

$$d = \inf\{\, |x-y| : x \in K \text{ and } y \in L \,\}.$$

This turns out to be a reasonable definition for the distance between K and L.
(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.
(b) Show that it is possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Solution. (a) The infimum is attained, and disjointness then forces it to be positive.
The set $\{|x-y| : x \in K,\ y \in L\}$ is nonempty and bounded below by 0, so d exists (Exercise 1.3.3). By the infimum version of Lemma 1.3.8 (Exercise 1.3.1(b)): no $d + 1/n$ is a lower bound, so pick for each $n \in \mathbf{N}$ points $x_n \in K$, $y_n \in L$ with

$$d \leq |x_n - y_n| < d + \tfrac{1}{n}.$$

By Definition 3.3.1 pass to a subsequence along which $x_n \to x_0 \in K$, then to a further subsequence along which $y_n \to y_0 \in L$; the first limit survives the second passage by Theorem 2.5.2. Relabelling, the Algebraic Limit Theorem (Theorem 2.3.3) together with $\big||u| - |v|\big| \leq |u - v|$ gives

$$|x_0 - y_0| = \lim_n |x_n - y_n| = d,$$

the second equality by the squeeze above. Because $K \cap L = \emptyset$ we have $x_0 \neq y_0$, hence $d = |x_0 - y_0| > 0$.
(b) Take

$$K = \mathbf{N}, \qquad L = \{\, n + 1/n : n \geq 2 \,\}.$$

Consecutive points of K differ by 1 and consecutive points of L by $\big(n+1+\frac{1}{n+1}\big) - \big(n+\frac{1}{n}\big) = 1 - \frac{1}{n(n+1)} \geq \frac{5}{6}$, so each $V_{1/4}(y)$ meets each set in at most one point: neither set has a limit point, so both contain their limit points vacuously and are closed (Definition 3.2.7). They are disjoint, since $n + 1/n$ is not an integer for $n \geq 2$. Finally

$$0 \leq d \leq \big|(n + 1/n) - n\big| = 1/n \quad \text{for every } n \geq 2,$$

so $d = 0$ although $K \cap L = \emptyset$.

Problem 3.3.9. Follow these steps to prove the final implication in Theorem 3.3.8 (the Heine–Borel Theorem), which asserts the equivalence of (i) K is compact, (ii) K is closed and bounded, and (iii) every open cover for K has a finite subcover; the implication left open there is (ii) $\Rightarrow$ (iii).
Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let us assume that no finite subcover exists. Let I_0 be a closed interval containing K.
(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.
(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.
(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution. (a) Bisect repeatedly. Since K is bounded, a closed interval $I_0 \supseteq K$ exists, and $I_0 \cap K = K$ is not finitely covered by hypothesis. Given I_n with $I_n \cap K$ not finitely coverable, write $I_n = A \cup B$ where A, B are the two closed halves of I_n, so that

$$I_n \cap K = (A \cap K) \cup (B \cap K).$$

If both $A \cap K$ and $B \cap K$ admitted finite subcovers, the union of those two finite collections would be a finite subcover of $I_n \cap K$. Hence at least one half does not; call it I_{n+1}. By construction $I_0 \supseteq I_1 \supseteq \cdots$ and

$$|I_n| = \frac{|I_0|}{2^n} \longrightarrow 0.$$

(b) Each $I_n \cap K \neq \emptyset$ (the empty set is covered by the empty finite subcollection), so pick $x_n \in I_n \cap K$. The Nested Interval Property (Theorem 1.4.1) supplies $x \in \bigcap_{n=0}^{\infty} I_n$, and since $x, x_n \in I_n$,

$$|x_n - x| \leq |I_n| \to 0, \qquad \text{so } (x_n) \to x.$$

The sequence $(x_n) \subseteq K$ is convergent, hence Cauchy (Theorem 2.6.2), so $x \in K$ by Theorem 3.2.8. Thus $x \in K$ and $x \in I_n$ for all n.
(c) Take O_{λ_0} from the cover with $x \in O_{\lambda_0}$. Openness gives $\epsilon > 0$ with $V_\epsilon(x) \subseteq O_{\lambda_0}$, and since $|I_n| \to 0$ we may fix n with $|I_n| < \epsilon$. Every $y \in I_n$ satisfies $|y - x| \leq |I_n| < \epsilon$ because $x \in I_n$, so

$$I_n \cap K \subseteq I_n \subseteq V_\epsilon(x) \subseteq O_{\lambda_0}.$$

The single set O_{λ_0} is therefore a finite subcover of $I_n \cap K$, contradicting (a). Hence a finite subcover of K exists after all.

Problem 3.3.10. Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine–Borel Theorem.
Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.
(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.
(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.
(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Solution. (a) $a \in S$, because $a \in [a, b] \subseteq \bigcup_\lambda O_\lambda$ puts a in some O_{λ_1}, and the single set O_{λ_1} covers $[a, a] = \{a\}$. Also $S \subseteq [a, b]$ is bounded above by b. So $s = \sup S$ exists by the Axiom of Completeness,

and $a \le s \le b$.
(b) Since $s \in [a,b]$, fix λ_0 with $s \in O_{\lambda_0}$ and then $\epsilon > 0$ with $V_\epsilon(s) \subseteq O_{\lambda_0}$. By Lemma 1.3.8 there is $x \in S$ with $x > s - \epsilon$, and by definition of S a finite subcover $O_{\lambda_1}, \ldots, O_{\lambda_n}$ of $[a,x]$. For any t with $x \le t < s + \epsilon$ and $t \le b$,

$$[a,t] \subseteq [a,x] \cup (s-\epsilon,\, s+\epsilon) \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n} \cup O_{\lambda_0},$$

since a point of $[a,t]$ exceeding x lies in $(s-\epsilon, s+\epsilon)$. Hence every such t belongs to S.
Taking $t = s$ gives $s \in S$. If $s < b$, then $t = \min\{b,\, s+\epsilon/2\}$ satisfies $x \le s < t \le b$ and $t < s+\epsilon$, so $t \in S$ with $t > s = \sup S$ — impossible. Therefore $s = b$, and $b = s \in S$ means $[a,b]$ has a finite subcover.
(c) Let K be closed and bounded and let $\{O_\lambda\}$ be an open cover of K. Boundedness gives $K \subseteq [a,b]$ for some $a \le b$, and closedness makes K^c open (Theorem 3.2.13). Then

$$\{O_\lambda : \lambda \in \Lambda\} \cup \{K^c\}$$

is an open cover of $[a,b]$, since every point of $[a,b]$ is either in K (hence in some O_λ) or in K^c. By (b) it has a finite subcover; delete K^c from that finite list if it appears. The remaining $O_{\lambda_1}, \ldots, O_{\lambda_n}$ still cover K, because no point of K was covered by K^c. That is a finite subcover of the original collection.

Problem 3.3.11. Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover. The sets are:
(a) $\mathbf{N}$.
(b) $\mathbf{Q} \cap [0,1]$.
(c) The Cantor set.
(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.
(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

Solution. Only (a), (b), (d) need covers: the Cantor set in (c) is an intersection of closed sets, hence closed (Theorem 3.2.14), and lies in $[0,1]$; the set in (e) is $\{1\} \cup \{(n-1)/n : n \ge 2\} \subseteq [0,1]$, whose only limit point is 1, an element of it. Both are closed and bounded, hence compact by Theorem 3.3.4.
(a) $\mathbf{N}$. Take

$$O_n = \left(n - \tfrac{1}{2},\ n + \tfrac{1}{2}\right), \qquad n \in \mathbf{N}.$$

Then $n \in O_n$, so $\{O_n\}$ covers $\mathbf{N}$, while $O_n \cap \mathbf{N} = \{n\}$. Any finite subcollection $O_{n_1}, \ldots, O_{n_k}$ therefore covers only the k points $n_1, \ldots, n_k$ and misses $\max\{n_1, \ldots, n_k\} + 1$.
(b) $\mathbf{Q} \cap [0,1]$. Let $\alpha = \sqrt{2}/2 \in (0,1)$, irrational because $2\alpha = \sqrt{2}$ is (Theorem 1.1.1), and set

$$O_n = (-\infty,\ \alpha - 1/n) \cup (\alpha + 1/n,\ \infty) = \{\, y : |y - \alpha| > 1/n \,\}.$$

Each O_n is open, and $\bigcup_n O_n = \mathbf{R} \setminus \{\alpha\} \supseteq \mathbf{Q} \cap [0,1]$, since any $y \ne \alpha$ lies in O_n once $1/n < |y - \alpha|$. The O_n increase, so a finite subcollection has union O_N for N the largest index. By the density of $\mathbf{Q}$ (Theorem 1.4.3) choose a rational

$$q \in \left(\alpha,\ \min\left\{1,\ \alpha + \tfrac{1}{N}\right\}\right),$$

a nonempty interval because $\alpha < 1$. Then $q \in \mathbf{Q} \cap [0,1]$ but $|q - \alpha| < 1/N$, so $q \notin O_N$. No finite subcover.
(d) Write $s_n = \sum_{k=1}^{n} 1/k^2$, so the set is $A = \{s_n : n \in \mathbf{N}\}$ and $s_1 < s_2 < s_3 < \cdots$. Put

$$O_n = \left(-\infty,\ \frac{s_n + s_{n+1}}{2}\right), \qquad n \in \mathbf{N}.$$

Since $s_j < \frac{s_n + s_{n+1}}{2}$ exactly when $j \le n$, the set O_n contains $s_1, \ldots, s_n$ and excludes $s_{n+1}, s_{n+2}, \ldots$; in particular $s_n \in O_n$ for every n, so $\{O_n\}$ covers A. The O_n are nested increasing, so any finite subcollection has union O_N, which omits $s_{N+1} \in A$. No finite subcover.

Problem 3.3.12. Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point.

Solution. If a bounded infinite set A had no limit point, the neighborhoods isolating its points would form an open cover with no finite subcover.
Having no limit point, A contains its limit points vacuously and so is closed (Definition 3.2.7); being closed and bounded, it admits a finite subcover from every open cover by the implication (ii) $\Rightarrow$ (iii) of Theorem 3.3.8, proved in Exercise 3.3.9 by bisection and the Nested Interval Property (Theorem 1.4.1) with no appeal to Bolzano–Weierstrass.
Since no $a \in A$ is a limit point of A, Definition 3.2.4 supplies $\epsilon_a > 0$ with $V_{\epsilon_a}(a) \cap A = \{a\}$, and $a \in V_{\epsilon_a}(a)$ makes $\{V_{\epsilon_a}(a) : a \in A\}$ an open cover of A. Writing $V_i = V_{\epsilon_{a_i}}(a_i)$ for the n members of a finite subcover, each $V_i \cap A = \{a_i\}$, so

$$A = \bigcup_{i=1}^{n} (V_i \cap A) = \{a_1, \dots, a_n\}$$

is finite, contradicting the hypothesis. Hence A has a limit point.

Problem 3.3.13. Let us call a set clompact if it has the property that every closed cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbf{R}$.

Solution. The clompact subsets of $\mathbf{R}$ are precisely the finite sets (including $\emptyset$).
Finite sets are clompact: if $A = \{a_1, \dots, a_n\}$ and $\{C_\lambda\}$ is any cover of A, pick λ_i with $a_i \in C_{\lambda_i}$; then

$$A \subseteq C_{\lambda_1} \cup \cdots \cup C_{\lambda_n},$$

a finite subcover (for $A = \emptyset$ the empty subcollection serves).
Infinite sets are not: the singletons $\{a\}$, $a \in A$, are closed (having no limit points, each contains them vacuously), and

$$A = \bigcup_{a \in A} \{a\}$$

is a closed cover of A. A finite subcollection $\{a_1\}, \dots, \{a_n\}$ has union $\{a_1, \dots, a_n\}$, which cannot contain the infinite set A. So no finite subcover exists.

3.5 Exercises 3.4.1–3.4.7

Problem 3.4.1. If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution. Always compact; not always perfect.
P is closed (Definition 3.4.1) and K is closed and bounded (Theorem 3.3.4), so $P \cap K$ is closed by Theorem 3.2.14 (ii) and bounded because $P \cap K \subseteq K$; Theorem 3.3.4 again makes it compact.
Perfection fails for $P = [0, 1]$ and $K = \{0\}$, the latter closed and bounded hence compact:

$$P \cap K = \{0\}, \qquad V_{1/2}(0) \cap \{0\} = \{0\},$$

so 0 is an isolated point of $P \cap K$.

Problem 3.4.2. Does there exist a perfect set consisting of only rational numbers?

Solution. No, apart from the vacuous $P = \emptyset$ (closed, with no isolated points).
A nonempty perfect set is uncountable by Theorem 3.4.3, while any $P \subseteq \mathbb{Q}$ is countable or finite by Theorem 1.5.7, since $\mathbb{Q}$ is countable by Theorem 1.5.6 (i). No set is both.

Problem 3.4.3. Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument. (Example 3.4.2 shows the Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ is perfect; C_n is the union of 2^n closed intervals each of length $1/3^n$, and it remains to show that an arbitrary $x \in C$ is not isolated.)
(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.
(b) Finish the proof by showing that for each $n \in \mathbb{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| \leq 1/3^n$.

Solution. Take x_n to be an endpoint, other than x itself, of the component interval of C_n containing x. The one fact needed is the observation from Section 3.1: if y is an endpoint of one of the closed intervals making up C_n, then y is again an endpoint of an interval of C_{n+1} (middle thirds removed are open, so endpoints are never deleted), whence $y \in C_m$ for all m and so $y \in C$.
(a) $C_1 = [0, 1/3] \cup [2/3, 1]$, so x lies in one of these two closed intervals, say $I = [a, b]$ with $b - a = 1/3$. Both a and b lie in C by the observation above, and at most one of them equals x; let x_1 be one that does not. Then

$$x_1 \in C \cap C_1, \qquad x_1 \neq x, \qquad |x - x_1| \leq b - a = \tfrac{1}{3},$$

the last inequality because $x, x_1 \in I$.
(b) Identical for every n. Since $x \in C \subseteq C_n$, the point x lies in one of the 2^n component intervals $I = [a, b]$ of C_n, and $b - a = 1/3^n$. Both endpoints belong to C, so choosing $x_n \in \{a, b\}$ with $x_n \neq x$ gives

$$x_n \in C \cap C_n, \qquad x_n \neq x, \qquad |x - x_n| \leq \tfrac{1}{3^n}.$$

Hence (x_n) is a sequence in $C \setminus \{x\}$ with $|x - x_n| \leq 1/3^n \to 0$, so $(x_n) \to x$ and x is a limit point of C (Theorem 3.2.5), i.e. not isolated. As $x \in C$ was arbitrary and C is closed, C is perfect.

Problem 3.4.4. Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle fourth from each component.
(a) Is the resulting set compact? Perfect?
(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Solution. Compact and perfect; length 0; dimension $\log 2/\log(8/3) \approx 0.707$.
Write $D_0 = [0, 1]$ and let D_{n+1} arise from D_n by deleting from each component the open middle fourth of that component. Deleting the middle quarter of an interval of length L leaves two closed intervals of length $\frac{3}{8}L$ each, so

$$D_1 = \left[0, \tfrac{3}{8}\right] \cup \left[\tfrac{5}{8}, 1\right],$$

and inductively D_n is a union of 2^n closed intervals each of length $(3/8)^n$. Set $D = \bigcap_{n=0}^{\infty} D_n$.
(a) Each D_n is a finite union of closed intervals, hence closed, and D is closed by Theorem 3.2.14 (ii); $D \subseteq [0, 1]$ is bounded. By Theorem 3.3.4, D is compact.
D is perfect by Exercise 3.4.3 with $1/3^n$ replaced by $(3/8)^n$: the removed middles are open, so endpoints of components of D_n survive every later stage and lie in D, and taking x_n to be the endpoint other than x of the component $[a, b]$ of D_n containing x gives

$$x_n \in D \setminus \{x\}, \qquad |x - x_n| \leq b - a = \left(\tfrac{3}{8}\right)^n \to 0.$$

So x is a limit point of D (Theorem 3.2.5); no point of D is isolated, and D is closed.
(b) At stage $n \geq 1$ we delete one open interval of length $\frac{1}{4}(3/8)^{n-1}$ from each of the 2^{n-1} components of D_{n-1}. The total length removed is

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{n-1} \cdot \tfrac{1}{4} \left(\tfrac{3}{8}\right)^{n-1} &= \tfrac{1}{4} \sum_{n=1}^{\infty} \left(\tfrac{3}{4}\right)^{n-1} \\ &= \tfrac{1}{4} \cdot \frac{1}{1 - 3/4} = 1,\end{aligned}$$

so the length of D is $1 - 1 = 0$.

For the dimension, magnify by 8/3, the reciprocal of a component's length: D_0 becomes $[0, 8/3]$ and deleting its middle fourth leaves $[0,1] \cup [5/3, 8/3]$, so the magnified set is 2 copies of D. Following Figure 3.3, the dimension x satisfies

$$2 = \left(\tfrac{8}{3}\right)^x, \qquad x = \frac{\log 2}{\log 8 - \log 3} = \frac{\log 2}{3\log 2 - \log 3} \approx 0.707.$$

Problem 3.4.5. Let A and B be nonempty subsets of $\mathbb{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution. A point of $\overline{A} \cap B$ would have a neighborhood inside V meeting $A \subseteq U$, contradicting $U \cap V = \emptyset$. In detail, suppose $x \in \overline{A} \cap B$. Since $x \in B \subseteq V$ and V is open, Definition 3.2.1 supplies $\epsilon > 0$ with $V_\epsilon(x) \subseteq V$. Since $\overline{A} = A \cup L$ with L the set of limit points of A (Definition 3.2.11), either $x \in A$ or x is a limit point of A; in both cases $V_\epsilon(x) \cap A \neq \emptyset$. Picking a in that intersection,

$$a \in A \subseteq U \quad \text{and} \quad a \in V_\epsilon(x) \subseteq V,$$

so $a \in U \cap V = \emptyset$, a contradiction. Hence $\overline{A} \cap B = \emptyset$.
Interchanging the roles of (A, U) and (B, V) gives $A \cap \overline{B} = \emptyset$. By Definition 3.4.4, A and B are separated.

Problem 3.4.6. Prove Theorem 3.4.6: a set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Solution. Both directions run through Theorem 3.2.5, which trades the statement that x is a limit point of A for a sequence (x_n) in A with $x_n \neq x$ and $(x_n) \to x$.
($\Rightarrow$) Let E be connected and let $E = A \cup B$ with A, B nonempty and disjoint. If A and B were separated, E would be disconnected by Definition 3.4.4; so they are not, meaning

$$\overline{A} \cap B \neq \emptyset \quad \text{or} \quad A \cap \overline{B} \neq \emptyset.$$

Say $x \in \overline{A} \cap B$ (the other case is symmetric). Then $x \in B$, and since $A \cap B = \emptyset$ we have $x \notin A$; as $\overline{A} = A \cup L$ (Definition 3.2.11), x must be a limit point of A. By Theorem 3.2.5 there is a sequence (x_n) contained in A with $(x_n) \to x$, and $x \in B$ is an element of the other set, as required.
($\Leftarrow$) Contrapositive: assume E is disconnected, so $E = A \cup B$ with A, B nonempty and separated. Separated sets are disjoint, since

$$A \cap B \subseteq \overline{A} \cap B = \emptyset,$$

so A, B is an admissible partition. Suppose (x_n) lies in A with $(x_n) \to x$ and $x \in B$. Then $x \notin A$, so every neighborhood $V_\epsilon(x)$ contains some $x_n \in A$ with $x_n \neq x$, making x a limit point of A; hence $x \in \overline{A} \cap B = \emptyset$, absurd. Symmetrically no sequence in B converges to a point of A, so this partition admits no such sequence.

Problem 3.4.7. A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.
(a) Show that $\mathbb{Q}$ is totally disconnected.
(b) Is the set of irrational numbers totally disconnected?

Solution. Cut at a point of the complementary set: an irrational for $\mathbb{Q}$, a rational for $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. Both sets are totally disconnected.
(a) Let $x, y \in \mathbb{Q}$ be distinct, labelled so that $x < y$. By Corollary 1.4.4 there is an irrational α with $x < \alpha < y$. Put

$$A = \mathbb{Q} \cap (-\infty, \alpha), \qquad B = \mathbb{Q} \cap (\alpha, \infty).$$

Since $\alpha \notin \mathbb{Q}$, no rational is omitted, so $\mathbb{Q} = A \cup B$, and $x \in A$, $y \in B$ makes both nonempty. The sets $U = (-\infty, \alpha)$ and $V = (\alpha, \infty)$ are open and disjoint with $A \subseteq U$ and $B \subseteq V$, so A and B are separated by Exercise 3.4.5.
(b) Yes. Let $x, y \in \mathbb{I}$ be distinct, labelled so that $x < y$. By Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$) there is a rational r with $x < r < y$. Put

$$A = \mathbb{I} \cap (-\infty, r), \qquad B = \mathbb{I} \cap (r, \infty).$$

Since $r \notin \mathbb{I}$ we have $\mathbb{I} = A \cup B$, with $x \in A$ and $y \in B$; and $A \subseteq (-\infty, r)$, $B \subseteq (r, \infty)$, two disjoint open sets, so A and B are separated by Exercise 3.4.5.

3.6 Exercises 3.4.8–3.5.5

Problem 3.4.8. Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7. (Recall from that exercise: a set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.) Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.
(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.
(b) Show that C is totally disconnected.

Solution. (a) Any N with $3^{-N} < \epsilon$ works, and one exists because $3^{-N} \leq 1/N$ while the Archimedean Property (Theorem 1.4.2) supplies N with $1/N < \epsilon$. Indeed, by the construction in Section 3.1, C_N is a union of 2^N pairwise disjoint closed intervals, each of length 3^{-N}; if x and y both belonged to one of them, say $[a, b]$, then

$$\epsilon = y - x \leq b - a = 3^{-N} < \epsilon,$$

which is absurd.
(b) Fix $x, y \in C$ with $x < y$ and take N as in (a). Let $[a, b]$ and $[c, d]$ be the intervals of C_N containing x and y; by (a) they are distinct, hence disjoint, and $d \geq y > x \geq a$ rules out $d < a$, so $c > b$. Among the finitely many intervals of C_N, let b' be the smallest left endpoint exceeding b (there is one, namely c), so $b < b' \leq c \leq y$. Then

$$(b, b') \cap C_N = \emptyset,$$

since an interval $[p, q]$ of C_N meeting (b, b') has $p \leq b$ by minimality of b' and $q > b$, so $b \in [p, q]$ and disjointness forces $[p, q] = [a, b]$, i.e. $q = b$, a contradiction. Choose any $z \in (b, b')$. Then $x \leq b < z < y$ and $z \notin C_N \supseteq C$. Now

$$A = C \cap (-\infty, z), \qquad B = C \cap (z, \infty)$$

are nonempty ($x \in A$, $y \in B$), satisfy $A \cup B = C$ precisely because $z \notin C$, and are separated by Exercise 3.4.5, being contained in the disjoint open sets $(-\infty, z)$ and (z, ∞). So C is totally disconnected.

Problem 3.4.9. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbf{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$, and let $F = O^c$.
(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.
(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)
(c) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

Solution. (a) O is open, being a union of the open sets $V_{\epsilon_n}(r_n)$ (Theorem 3.2.3(i)), so $F = O^c$ is closed by Theorem 3.2.13. It contains no rational number: every $q \in \mathbf{Q}$ equals r_n for some n, and $r_n \in V_{\epsilon_n}(r_n) \subseteq O$.
For $F \neq \emptyset$ it suffices to show $[0, 3] \not\subseteq O$. Suppose it were. Since $[0, 3]$ is compact, the Heine-Borel

Theorem (Theorem 3.3.8) yields a finite set $S \subseteq \mathbf{N}$ with $[0,3] \subseteq \bigcup_{n \in S} V_{\epsilon_n}(r_n)$, a union of finitely many open intervals of total length

$$\sum_{n \in S} 2\epsilon_n \;\le\; \sum_{n=1}^{\infty} \frac{2}{2^n} \;=\; 2.$$

This contradicts the following fact, applied with $[a,b] = [0,3]$: for nonempty open intervals (a_k, b_k),

$$[a,b] \subseteq \bigcup_{k=1}^{m} (a_k, b_k) \quad \Longrightarrow \quad b - a < \sum_{k=1}^{m} (b_k - a_k).$$

Induct on m. For $m = 1$, $a_1 < a \le b < b_1$ gives $b - a < b_1 - a_1$. For $m \ge 2$, relabel so that $b \in (a_m, b_m)$. If $a_m < a$ we are done as in the base case. Otherwise $a \le a_m \le b$, and no point of $[a, a_m]$ lies in (a_m, b_m), so $[a, a_m]$ is covered by the other $m-1$ intervals; the induction hypothesis and $b - a_m < b_m - a_m$ give

$$b - a \;=\; (a_m - a) + (b - a_m) \;<\; \sum_{k=1}^{m} (b_k - a_k).$$

(b) No, and yes. Any nonempty open interval (c,d) contains a rational (Theorem 1.4.3), and that rational lies in O, so $(c,d) \nsubseteq F$; thus F contains no nonempty open interval. For total disconnectedness, take $x, y \in F$ with $x < y$ and pick a rational $r \in (x,y)$, again by Theorem 1.4.3. Then $r \notin F$, so

$$A = F \cap (-\infty, r), \qquad B = F \cap (r, \infty)$$

are nonempty ($x \in A$, $y \in B$), satisfy $A \cup B = F$, and are separated because they sit inside the disjoint open sets $(-\infty, r)$ and (r, ∞) (Exercise 3.4.5).

(c) No: the construction does not determine the answer, because some enumerations leave F with an isolated point and hence not perfect. Since F is closed, perfection asks only that no point be isolated (Definition 3.4.1). Fix an irrational x and reserve in advance, for every $k \ge 1$, rationals

$$\begin{aligned} r_{3k} &\quad \text{with} \quad 2^{-3k} < x - r_{3k} < (1.1)2^{-3k}, \\ r_{3k+1} &\quad \text{with} \quad 2^{-(3k+1)} < r_{3k+1} - x < (1.1)2^{-(3k+1)}, \end{aligned}$$

these being distinct, since the intervals they are drawn from are pairwise disjoint and lie on opposite sides of x. To the remaining indices n assign rationals outside the reserved list, at each step taking the first unused such q (in some fixed master list) with $|x - q| \ge 2^{-n}$; every rational is eventually used, since only finitely many precede a given q and q is eligible at every step with $2^{-n} \le |x - q|$. Then $x \notin O$, as $|x - r_n| \ge \epsilon_n$ for every n. Writing $\rho = 2^{-3k}$, the choice above gives $V_\rho(r_{3k}) \supseteq [x - 2\rho, x - \rho/10]$, and consecutive members of this chain overlap because $2 \cdot 2^{-3(k+1)} = \rho/4 > \rho/10$; hence the chain covers $[x - 1/4, x)$. The chain of the r_{3k+1} covers $(x, x + 1/8]$ in the same way. So $F \cap V_{1/8}(x) = \{x\}$ and F is not perfect for this enumeration.

We can, however, always build a nonempty perfect set of irrationals by a Cantor-style construction. Let $K_0 = [0,1]$, and suppose K_{n-1} is a union of 2^{n-1} pairwise disjoint nondegenerate closed intervals. Inside each such interval I choose two disjoint nondegenerate closed subintervals, each of length less than 2^{-n} and neither containing r_n (possible because I minus the single point r_n always contains two such intervals), and let K_n be the union of the resulting 2^n intervals. Put $K = \bigcap_{n=0}^{\infty} K_n$. Each K_n is nonempty, compact and $K_{n+1} \subseteq K_n$, so $K \neq \emptyset$ by the Nested Compact Set Property (Theorem 3.3.5), and K is closed by Theorem 3.2.14(ii). No rational survives, since $r_n \notin K_n$. Finally, let $p \in K$ and $\delta > 0$, and pick $n \ge 1$ with $2^{-n} < \delta$. The interval I of K_n containing p has length less than δ, so $I \subseteq V_\delta(p)$, and one of the two stage-$(n+1)$ intervals $J \subseteq I$ omits p. Following a nested chain of intervals downward from J and applying the Nested Interval Property (Theorem 1.4.1) produces a point $y \in K \cap J$, so $y \in K \cap V_\delta(p)$ with $y \neq p$. Hence K has no isolated points: K is a nonempty perfect set of irrational numbers.

Problem 3.5.1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Solution. De Morgan plus Theorem 3.2.13 (O is open if and only if O^c is closed).
($\Rightarrow$) If $A = \bigcap_{n=1}^\infty O_n$ with each O_n open (Definition 3.5.1), then

$$A^c = \Big(\bigcap_{n=1}^\infty O_n\Big)^c = \bigcup_{n=1}^\infty O_n^c,$$

and each O_n^c is closed by Theorem 3.2.13, so A^c is a countable union of closed sets, i.e. an F_σ set.
($\Leftarrow$) If $A^c = \bigcup_{n=1}^\infty F_n$ with each F_n closed, then

$$A = (A^c)^c = \bigcap_{n=1}^\infty F_n^c,$$

and each F_n^c is open by the second half of Theorem 3.2.13, so A is a G_δ set.

Problem 3.5.2. Replace each blank with the word *finite* or *countable*, depending on which is more appropriate.
(a) The ___ union of F_σ sets is an F_σ set.
(b) The ___ intersection of F_σ sets is an F_σ set.
(c) The ___ union of G_δ sets is a G_δ set.
(d) The ___ intersection of G_δ sets is a G_δ set.

Solution. (a) **countable**, (b) **finite**, (c) **finite**, (d) **countable**.
(a) Write $A_k = \bigcup_{n=1}^\infty F_{k,n}$ with each $F_{k,n}$ closed, $k \in \mathbf{N}$. Then

$$\bigcup_{k=1}^\infty A_k = \bigcup_{(k,n)\in\mathbf{N}\times\mathbf{N}} F_{k,n},$$

a union indexed by a countable set (Theorem 1.5.8 (ii)), hence an F_σ set.
(b) Let $A_k = \bigcup_{n=1}^\infty F_{k,n}$ for $k = 1, \dots, m$. Replacing $F_{k,n}$ by $E_{k,n} = F_{k,1} \cup \cdots \cup F_{k,n}$ (closed by Theorem 3.2.14 (i)) changes no union and makes each family increasing in n. Then

$$\bigcap_{k=1}^m A_k = \bigcup_{n=1}^\infty \bigcap_{k=1}^m E_{k,n} :$$

$\supseteq$ is clear, and if $x \in A_k$ for every k, choose n_k with $x \in E_{k,n_k}$ and set $n = \max\{n_1, \dots, n_m\}$, which exists because there are finitely many k; monotonicity gives $x \in E_{k,n}$ for all k. Each $\bigcap_{k=1}^m E_{k,n}$ is closed by Theorem 3.2.14 (ii), so the right-hand side is an F_σ set.
Countably many will not do in (b): each

$$\mathbf{R} \setminus \{q\} = \bigcup_{n=2}^\infty \Big(\big[q-n,\, q-\tfrac{1}{n}\big] \cup \big[q+\tfrac{1}{n},\, q+n\big]\Big)$$

is F_σ, yet $\mathbf{I} = \bigcap_{q\in\mathbf{Q}}(\mathbf{R}\setminus\{q\})$ is not (Exercise 3.5.6).
(c), (d) Take complements. By Exercise 3.5.1, B_k is G_δ exactly when B_k^c is F_σ, and

$$\Big(\bigcup_k B_k\Big)^c = \bigcap_k B_k^c, \qquad \Big(\bigcap_k B_k\Big)^c = \bigcup_k B_k^c.$$

So (c) is the complementary statement of (b) (finite) and (d) is the complementary statement of (a) (countable). Countably many sets fail in (c) for the dual reason: $\mathbf{Q} = \bigcup_{q\in\mathbf{Q}}\{q\}$ with $\{q\} = \bigcap_{n=1}^\infty (q - \frac{1}{n}, q + \frac{1}{n})$ a G_δ set, yet $\mathbf{Q}$ is not G_δ (Exercise 3.5.6).

Problem 3.5.3. (This exercise has already appeared as Exercise 3.2.15.)
(a) Show that a closed interval $[a, b]$ is a G_δ set.
(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.
(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set.

Solution. (a) $[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n},\ b + \frac{1}{n}\right)$, an intersection of open sets. Indeed $[a,b]$ lies in every one of them, and if x is in every one then $a - \frac{1}{n} < x < b + \frac{1}{n}$ for all n, so $a \le x \le b$ by the Order Limit Theorem (Theorem 2.3.4) applied to $a - \frac{1}{n} \to a$ and $b + \frac{1}{n} \to b$.
(b) For the G_δ form,

$$(a,b] = \bigcap_{n=1}^{\infty} \left(a,\ b + \tfrac{1}{n}\right),$$

by the same squeeze on the right endpoint. For the F_σ form,

$$(a,b] = \bigcup_{n=1}^{\infty} \left[a + \tfrac{b-a}{n},\ b\right] :$$

each such closed interval sits inside $(a,b]$, and given $x \in (a,b]$ we have $x - a > 0$, so the Archimedean Property (Theorem 1.4.2) supplies n with $\frac{b-a}{n} \le x - a$, putting x in the n-th interval.
(c) $\mathbf{Q}$ is countable (Theorem 1.5.6 (i)), so $\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}$ is a countable union of the closed sets $\{q\}$ (a single point has no limit points, so it vacuously contains them, Definition 3.2.7), hence F_σ. Since $\mathbf{I} = \mathbf{Q}^c$, Exercise 3.5.1 makes $\mathbf{I}$ a G_δ set. Explicitly, enumerating $\mathbf{Q} = \{q_1, q_2, q_3, \ldots\}$,

$$\mathbf{I} = \bigcap_{n=1}^{\infty} \left(\mathbf{R} \setminus \{q_n\}\right).$$

Problem 3.5.4. This exercise completes the proof of Theorem 3.5.2: if $\{G_1, G_2, G_3, \ldots\}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.
Starting with $n = 1$, inductively construct a nested sequence of *closed* intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Solution. Build each closed I_{n+1} strictly inside the *interior* of I_n, giving it half the radius of the neighborhood that produced it; that halving is what pulls the endpoints inside, and it is the special attention the endpoints require.
Base case: G_1 is dense, hence nonempty, so pick $x_1 \in G_1$; openness (Definition 3.2.1) gives $\varepsilon_1 > 0$ with $(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \subseteq G_1$. Set

$$I_1 = \left[x_1 - \tfrac{\varepsilon_1}{2},\ x_1 + \tfrac{\varepsilon_1}{2}\right] \subseteq G_1.$$

Inductive step: suppose $I_n = [a_n, b_n]$ with $a_n < b_n$ has been built. Since G_{n+1} is dense, there is a point $x_{n+1} \in G_{n+1}$ with $a_n < x_{n+1} < b_n$. The set $G_{n+1} \cap (a_n, b_n)$ is open (Theorem 3.2.3 (ii)) and contains x_{n+1}, so there is $\varepsilon_{n+1} > 0$ with

$$(x_{n+1} - \varepsilon_{n+1},\ x_{n+1} + \varepsilon_{n+1}) \subseteq G_{n+1} \cap (a_n, b_n).$$

Set $I_{n+1} = \left[x_{n+1} - \frac{\varepsilon_{n+1}}{2},\ x_{n+1} + \frac{\varepsilon_{n+1}}{2}\right]$, so

$$I_{n+1} \subseteq G_{n+1} \qquad \text{and} \qquad I_{n+1} \subseteq (a_n, b_n) \subseteq I_n.$$

The I_n are nonempty closed bounded intervals with $I_1 \supseteq I_2 \supseteq \cdots$, so the Nested Interval Property (Theorem 1.4.1) yields a point

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

For every n we have $x \in I_n \subseteq G_n$, so $x \in \bigcap_{n=1}^{\infty} G_n$ and the intersection is not empty.

Problem 3.5.5. Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbf{N}$, F_n is a closed set containing no nonempty open intervals.

Solution. Pass to complements and apply Theorem 3.5.2.
Assume such a decomposition exists and set $G_n = F_n^c$, which is open by Theorem 3.2.13.
Each G_n is dense: given $a < b$, the interval (a, b) is a nonempty open interval, so $(a, b) \not\subseteq F_n$ by hypothesis; any $x \in (a, b) \setminus F_n$ is a point of G_n with $a < x < b$.
Theorem 3.5.2 applies to the countable collection $\{G_1, G_2, G_3, \ldots\}$ of dense open sets and gives $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. But De Morgan turns the assumed decomposition into

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} F_n^c = \Big(\bigcup_{n=1}^{\infty} F_n\Big)^c = \mathbf{R}^c = \emptyset.$$

3.7 Exercises 3.5.6–3.5.10

Problem 3.5.6. Show how the previous exercise implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Solution. An F_σ decomposition of $\mathbf{I}$ would combine with the one of $\mathbf{Q}$ to split $\mathbf{R}$ in the manner forbidden by Exercise 3.5.5.
Suppose $\mathbf{I} = \bigcup_{n=1}^{\infty} F_n$ with each F_n closed. No F_n contains a nonempty open interval: any such interval (a, b) contains a rational by Theorem 1.4.3 (Density of $\mathbf{Q}$ in $\mathbf{R}$), while $F_n \subseteq \mathbf{I}$. Enumerate $\mathbf{Q} = \{q_1, q_2, q_3, \ldots\}$ (Theorem 1.5.6 (i)); each singleton $\{q_k\}$ is closed and plainly contains no nonempty open interval. Then

$$\mathbf{R} = \mathbf{I} \cup \mathbf{Q} = \Big(\bigcup_{n=1}^{\infty} F_n\Big) \cup \Big(\bigcup_{k=1}^{\infty} \{q_k\}\Big)$$

exhibits $\mathbf{R}$ as a countable union (Theorem 1.5.8 (ii)) of closed sets none of which contains a nonempty open interval, contradicting Exercise 3.5.5. Hence $\mathbf{I}$ is not F_σ.
If $\mathbf{Q}$ were G_δ, then $\mathbf{Q}^c = \mathbf{I}$ would be F_σ by Exercise 3.5.1, which we have just ruled out.

Problem 3.5.7. Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

Solution. Take

$$A = \big(\mathbf{Q} \cap (0, \infty)\big) \ \cup \ \big(\mathbf{I} \cap (-\infty, 0)\big),$$

the rationals to the right of the origin together with the irrationals to the left.
Lemma: neither $\mathbf{I} \cap (-\infty, 0)$ nor $\mathbf{I} \cap (0, \infty)$ is an F_σ set.
Reflection $x \mapsto -x$ carries $V_\varepsilon(x)$ onto $V_\varepsilon(-x)$, so it matches limit points with limit points and $-F = \{-x : x \in F\}$ is closed whenever F is (Definition 3.2.7); also $-\mathbf{I} = \mathbf{I}$. So if $\mathbf{I} \cap (-\infty, 0) = \bigcup_{n=1}^{\infty} F_n$ with each F_n closed, then $\mathbf{I} \cap (0, \infty) = \bigcup_{n=1}^{\infty} (-F_n)$, and since $0 \notin \mathbf{I}$,

$$\mathbf{I} = \bigcup_{n=1}^{\infty} F_n \ \cup \ \bigcup_{n=1}^{\infty} (-F_n)$$

would be F_σ, contradicting Exercise 3.5.6. Reflecting once more gives the same conclusion for $\mathbf{I} \cap (0, \infty)$.
The set A is not F_σ. The ray $(-\infty, 0) = \bigcup_{n=1}^{\infty} \big[-n, -\frac{1}{n}\big]$ is an F_σ set. If A were F_σ, then by the finite-intersection statement of Exercise 3.5.2 (b),

$$A \cap (-\infty, 0) = \mathbf{I} \cap (-\infty, 0)$$

would be F_σ, contradicting the Lemma.
The set A is not G_δ. If it were, A^c would be F_σ by Exercise 3.5.1. For $x > 0$ we have $x \in A$ precisely when $x \in \mathbf{Q}$, so $A^c \cap (0, \infty) = \mathbf{I} \cap (0, \infty)$. Since $(0, \infty) = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, n\right]$ is F_σ, Exercise 3.5.2 (b) would make $\mathbf{I} \cap (0, \infty)$ an F_σ set, again contradicting the Lemma.

Problem 3.5.8. Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

Solution. Both conditions say: every interval (a, b) with $a < b$ contains a point of $\overline{E}^c$.
Indeed, Definition 3.5.3 makes E nowhere-dense precisely when $\overline{E}$ contains no nonempty open interval, and the definition of density (p. 107) makes a set G dense precisely when every interval (a, b) with $a < b$ meets G. Hence

$$\begin{aligned} E \text{ is nowhere-dense} &\iff \neg\, \exists\, a < b \text{ with } (a, b) \subseteq \overline{E} \\ &\iff \forall\, a < b\ \exists\, x \in (a, b) \text{ with } x \notin \overline{E} \\ &\iff \forall\, a < b\ \exists\, x \in (a, b) \text{ with } x \in \overline{E}^c \\ &\iff \overline{E}^c \text{ is dense in } \mathbf{R}. \end{aligned}$$

The second step is just the negation of a containment: $(a, b) \not\subseteq \overline{E}$ says exactly that some point of (a, b) escapes $\overline{E}$.

Problem 3.5.9. Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.
(a) $A = \mathbf{Q} \cap [0, 5]$.
(b) $B = \{1/n : n \in \mathbf{N}\}$.
(c) the set of irrationals.
(d) the Cantor set.

Solution. (a) in between; (b) nowhere-dense; (c) dense; (d) nowhere-dense. In each case the verdict is read off the closure, using that G is dense iff $\overline{G} = \mathbf{R}$ and nowhere-dense iff $\overline{G}$ swallows no nonempty open interval (Definition 3.5.3).
(a) $\overline{A} = [0, 5]$: the containment $\subseteq$ holds because $[0, 5]$ is closed (Example 3.2.9 (ii)) and $\overline{A}$ is the smallest closed set containing A (Theorem 3.2.12), while $\supseteq$ holds because each $x \in [0, 5]$ is a limit of rationals drawn from $[0, 5]$, by the density of $\mathbf{Q}$ (Theorem 1.4.3) with Theorem 3.2.5. Now $\overline{A} \supseteq (0, 5)$ rules out nowhere-dense and $\overline{A} \neq \mathbf{R}$ rules out dense.
(b) $\overline{B} = B \cup \{0\}$, since 0 is the only limit point of B (Example 3.2.9 (i)). Every element of $\overline{B}$ is rational, while Corollary 1.4.4 puts an irrational inside each interval (a, b) with $a < b$; so no such interval lies in $\overline{B}$, and B is nowhere-dense.
(c) Given $a < b$, Corollary 1.4.4 supplies an irrational t with $a < t < b$, which is the definition of dense verbatim; so the set $\mathbf{I}$ of irrationals has $\overline{\mathbf{I}} = \mathbf{R}$.
(d) $C = \bigcap_{n \geq 0} C_n$ is an arbitrary intersection of closed sets, hence closed (Theorem 3.2.14 (ii)), so $\overline{C} = C$. Suppose $(a, b) \subseteq C$ with $a < b$. For every $n \in \mathbf{N}$ we then have $(a, b) \subseteq C_n$, a union of 2^n closed intervals of length 3^{-n}, any two of which are separated by a deleted open interval; an interval meeting two of these components would contain a point of the gap between them, so (a, b) sits inside one component and $b - a \leq 3^{-n} \leq 1/n$. Taking the n that Theorem 1.4.2 (ii) provides for the positive number $b - a$ gives $1/n < b - a$, a contradiction. Hence $\overline{C}$ contains no nonempty open interval.

Problem 3.5.10. This exercise completes the proof of Theorem 3.5.4 (Baire's Theorem): the set of real numbers $\mathbf{R}$ cannot be written as the countable union of nowhere-dense sets. The proof given in the text begins as follows. For contradiction, assume that $E_1, E_2, E_3, \ldots$ are each nowhere-dense and satisfy

$$\mathbf{R} = \bigcup_{n=1}^{\infty} E_n.$$

Finish the proof by finding a contradiction to the results in this section.

Solution. Take $G_n = \overline{E_n}^{\,c}$ and feed the collection $\{G_1, G_2, G_3, \dots\}$ into Theorem 3.5.2. Its three hypotheses hold: the collection is countable, being indexed by $\mathbf{N}$; each G_n is open, because $\overline{E_n}$ is a closed set (Theorem 3.2.12) and the complement of a closed set is open (Theorem 3.2.13); and each G_n is dense in $\mathbf{R}$, because E_n is nowhere-dense and Exercise 3.5.8 identifies that with density of the complement of $\overline{E_n}$.
Theorem 3.5.2 therefore supplies a point $x \in \bigcap_{n=1}^{\infty} G_n$. For this x,

$$\begin{aligned} x \in \bigcap_{n=1}^{\infty} G_n &\implies x \notin \overline{E_n} \text{ for every } n \in \mathbf{N} \\ &\implies x \notin E_n \text{ for every } n \in \mathbf{N} \\ &\implies x \notin \bigcup_{n=1}^{\infty} E_n = \mathbf{R}, \end{aligned}$$

the middle step because $E_n \subseteq \overline{E_n}$ (Definition 3.2.11). But x is a real number, so $x \in \mathbf{R}$. This contradiction refutes the assumption that $\mathbf{R}$ is a countable union of nowhere-dense sets.

4 Functional Limits and Continuity

Contents

4.1 Exercises 4.2.1–4.2.7

Problem 4.2.1. (a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.
(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.
(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

Solution. (a) Throughout, $f, g : A \to \mathbf{R}$, c is a limit point of A, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$. Let $(x_n) \subseteq A$ be arbitrary with $x_n \neq c$ and $x_n \to c$. Applying the forward implication (i) $\Rightarrow$ (ii) of Theorem 4.2.3 to f and then to g gives $f(x_n) \to L$ and $g(x_n) \to M$, so by the Algebraic Limit Theorem for sequences (Theorem 2.3.3 (ii)),

$$(f+g)(x_n) = f(x_n) + g(x_n) \longrightarrow L + M.$$

The sequence (x_n) was arbitrary, so the reverse implication (ii) $\Rightarrow$ (i) of Theorem 4.2.3, applied to the function $f+g$ and the candidate limit $L+M$, yields $\lim_{x\to c}[f(x)+g(x)] = L+M$.
(b) Let $\epsilon > 0$. Definition 4.2.1 applied to f with the challenge $\epsilon/2$ produces $\delta_1 > 0$ such that $0 < |x-c| < \delta_1$ (and $x \in A$) implies $|f(x)-L| < \epsilon/2$; applied to g with the same challenge it produces $\delta_2 > 0$ with $0 < |x-c| < \delta_2$ implying $|g(x)-M| < \epsilon/2$. Set $\delta = \min\{\delta_1, \delta_2\} > 0$. For $x \in A$ with $0 < |x-c| < \delta$ both estimates are in force, so

$$\begin{aligned}|(f+g)(x) - (L+M)| &\leq |f(x)-L| + |g(x)-M| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

(c) Sequentially: with (x_n) as in (a), Theorem 4.2.3 gives $f(x_n) \to L$ and $g(x_n) \to M$, and Theorem 2.3.3 (iii) gives $f(x_n)g(x_n) \to LM$; since (x_n) was arbitrary, Theorem 4.2.3 returns $\lim_{x\to c} f(x)g(x) = LM$.
Directly: the algebraic identity

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\leq |f(x)|\,|g(x)-M| + |M|\,|f(x)-L|\end{aligned}$$

reduces the problem to bounding $|f(x)|$. Choose $\delta_0 > 0$ so that $0 < |x-c| < \delta_0$ implies $|f(x)-L| < 1$, whence $|f(x)| < |L|+1$ there. Now let $\epsilon > 0$ and choose $\delta_1, \delta_2 > 0$ with

$$|g(x)-M| < \frac{\epsilon}{2(|L|+1)}, \qquad |f(x)-L| < \frac{\epsilon}{2(|M|+1)}$$

whenever $0 < |x-c| < \delta_1$, respectively $0 < |x-c| < \delta_2$. With $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and $0 < |x-c| < \delta$,

$$\begin{aligned}|f(x)g(x) - LM| &< (|L|+1)\frac{\epsilon}{2(|L|+1)} + |M|\frac{\epsilon}{2(|M|+1)} \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

Problem 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.
(a) $\lim_{x\to 3}(5x-6) = 9$, where $\epsilon = 1$.
(b) $\lim_{x\to 4}\sqrt{x} = 2$, where $\epsilon = 1$.
(c) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)
(d) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = .01$.

Solution. (a) $\delta = 1/5$. (b) $\delta = 3$. (c) $\delta = \pi - 3$. (d) $\delta = \pi - 3$.
In each part, $S = \{x : |f(x) - L| < \epsilon\}$ is computed exactly and δ is the distance from c to the nearest point outside S.
(a) $|(5x - 6) - 9| = 5|x - 3| < 1$ holds precisely when $|x - 3| < 1/5$, so $S = V_{1/5}(3)$ and the largest response is $\delta = 1/5$.
(b) $|\sqrt{x} - 2| < 1$ says $1 < \sqrt{x} < 3$, i.e. $S = (1, 9)$. The largest δ-neighborhood of 4 inside $(1, 9)$ has radius $\min\{4 - 1,\ 9 - 4\} = 3$, so $\delta = 3$.
(c) Since $[[x]]$ takes only integer values, $|[[x]] - 3| < 1$ forces $[[x]] = 3$, i.e. $S = [3, 4)$. Then

$$\delta = \min\{\pi - 3,\ 4 - \pi\} = \pi - 3 \approx 0.14159,$$

since $\pi - 3 < 1/2 < 4 - \pi$.
(d) The condition $|[[x]] - 3| < .01$ again forces the integer $[[x]]$ to equal 3, so $S = [3, 4)$ and $\delta = \pi - 3$ are unchanged: $[[x]]$ attains no value strictly between 3 and 4 for the smaller ϵ to exclude.

Problem 4.2.3. Review the definition of Thomae's function $t(x)$ from Section 4.1, namely

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \\ & \quad \text{in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

(a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.
(b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.
(c) Make an educated conjecture for $\lim_{x\to1} t(x)$, and use Definition 4.2.1B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$. Argue that all the points in this set are isolated.)

Solution. (a) Take

$$x_n = 1 + \frac{1}{n}, \qquad y_n = 1 + \frac{\sqrt{2}}{n}, \qquad z_n = 1 - \frac{1}{n+1},$$

all of which converge to 1 and none of whose terms equals 1.
(b) All three limits are 0. Indeed $x_n = (n+1)/n$ with $\gcd(n + 1, n) = 1$, so $t(x_n) = 1/n \to 0$; each y_n is irrational (a rational y_n would make $\sqrt{2} = n(y_n - 1)$ rational), so $t(y_n) = 0$ for every n; and $z_n = n/(n + 1)$ with $\gcd(n, n + 1) = 1$, so $t(z_n) = 1/(n + 1) \to 0$.
(c) Conjecture: $\lim_{x\to1} t(x) = 0$.
Fix $\epsilon > 0$ and put $S_\epsilon = \{x \in \mathbf{R} : t(x) \geq \epsilon\}$. If $x \in S_\epsilon$ then $t(x) > 0$, so x is rational; writing $x = m/n$ in lowest terms with $n > 0$ (and $0 = 0/1$), the value $t(x) = 1/n$ satisfies $1/n \geq \epsilon$, i.e.

$$n \leq 1/\epsilon.$$

Hence

$$S_\epsilon \cap (0, 2) \subseteq \left\{\tfrac{m}{n} : 1 \leq n \leq 1/\epsilon,\ 0 < m < 2n\right\},$$

a finite set: at most $\lfloor 1/\epsilon \rfloor$ denominators are admissible and fewer than $2n$ numerators for each. The same count on any bounded interval shows every point of S_ϵ is isolated.
Now let $F = \big(S_\epsilon \cap (0, 2)\big) \setminus \{1\}$, a finite set not containing 1, and set

$$\delta = \min\big(\{1\} \cup \{|x - 1| : x \in F\}\big) > 0,$$

a minimum of finitely many strictly positive numbers. If $x \in V_\delta(1)$ and $x \neq 1$, then $x \in (0, 2)$ and $x \notin F$, so $x \notin S_\epsilon$; that is, $t(x) < \epsilon$, and since $t \geq 0$ always, $t(x) \in V_\epsilon(0)$. This is exactly Definition 4.2.1B for $\lim_{x\to1} t(x) = 0$.

Problem 4.2.4. Consider the reasonable but erroneous claim that

$$\lim_{x\to 10} 1/[[x]] = 1/10.$$

(a) Find the largest δ that represents a proper response to the challenge of $\epsilon = 1/2$.
(b) Find the largest δ that represents a proper response to $\epsilon = 1/50$.
(c) Find the largest ϵ challenge for which there is no suitable δ response possible.

Solution. (a) $\delta = 8$. (b) $\delta = 1$. (c) $\epsilon = 1/90$.
Everything follows from one table. The domain of $f(x) = 1/[[x]]$ excludes $[0,1)$, and on the block $[n, n+1)$ with $n \in \mathbf{Z} \setminus \{0\}$ the function is constantly $1/n$, so the error is constantly

$$E(n) = \left|\frac{1}{n} - \frac{1}{10}\right| = \frac{|10-n|}{10\,|n|}.$$

The relevant values are

n	1	2	8	9	10	11	12	13
$E(n)$	$9/10$	$2/5$	$1/40$	$1/90$	0	$1/110$	$1/60$	$3/130$

and E decreases on $1 \le n \le 10$ and increases on $n \ge 10$, with $E(n) < 1/10$ for every $n \ge 10$.
(a) With $\epsilon = 1/2$ the blocks that fail are exactly those with $E(n) \ge 1/2$, and by the monotonicity just noted that means $n = 1$ alone: $E(1) = 9/10$ while $E(2) = 2/5 < 1/2$. So the admissible x form $[2, \infty)$, and the largest δ-neighborhood of 10 inside it is $V_8(10) = (2, 18)$. Thus $\delta = 8$; any larger δ admits a point of $[1, 2)$, where the error is $9/10$.
(b) With $\epsilon = 1/50 = 0.02$ the block $n = 9$ survives $(1/90 < 1/50)$ but $n = 8$ fails $(1/40 > 1/50)$, forcing $10 - \delta \ge 9$; on the right $n = 12$ survives $(1/60 < 1/50)$ but $n = 13$ fails $(3/130 > 1/50)$, forcing $10 + \delta \le 13$. Hence $\delta \le \min\{1, 3\} = 1$, and $\delta = 1$ does work since $V_1(10) = (9, 11)$ meets only the blocks $n = 9, 10$.
(c) Every punctured neighborhood of 10 contains points of $[9, 10)$, where the error is the fixed number $1/90$. So a δ response exists if and only if $\epsilon > 1/90$, and the challenges that cannot be met are exactly $0 < \epsilon \le 1/90$. The largest of these is

$$\epsilon = \frac{1}{90}.$$

Problem 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.
(a) $\lim_{x\to 2}(3x + 4) = 10$.
(b) $\lim_{x\to 0} x^3 = 0$.
(c) $\lim_{x\to 2}(x^2 + x - 1) = 5$.
(d) $\lim_{x\to 3} 1/x = 1/3$.

Solution. In each part let $\epsilon > 0$ be given; the displayed δ is the response demanded by Definition 4.2.1.
(a) Take $\delta = \epsilon/3$. If $0 < |x - 2| < \delta$ then

$$|(3x + 4) - 10| = 3|x - 2| < 3 \cdot \tfrac{\epsilon}{3} = \epsilon.$$

(b) Take $\delta = \epsilon^{1/3}$. If $0 < |x - 0| < \delta$ then

$$|x^3 - 0| = |x|^3 < \delta^3 = \epsilon.$$

(c) Take $\delta = \min\{1,\ \epsilon/6\}$. Factoring,

$$|(x^2 + x - 1) - 5| = |x^2 + x - 6| = |x + 3|\,|x - 2|,$$

and $|x - 2| < \delta \le 1$ gives $1 < x < 3$, hence $|x + 3| < 6$. Therefore $0 < |x - 2| < \delta$ implies

$$|x + 3|\,|x - 2| < 6 \cdot \tfrac{\epsilon}{6} = \epsilon.$$

(d) Take $\delta = \min\{1,\ 6\epsilon\}$. For $x \neq 0$,

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{3|x|},$$

and $|x-3| < \delta \leq 1$ gives $x > 2$, so $3|x| > 6$ and the fraction is at most $|x-3|/6$. Hence $0 < |x-3| < \delta$ implies

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \frac{6\epsilon}{6} = \epsilon.$$

Problem 4.2.6. Decide if the following claims are true or false, and give short justifications for each conclusion.
(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.
(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x) - 2]^2 = 3(L-2)^2$.
(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Solution. (a) True. (b) False. (c) True. (d) False.
(a) If $0 < \delta' \leq \delta$ and $0 < |x-a| < \delta'$, then $0 < |x-a| < \delta$, so the conclusion $|f(x) - L| < \epsilon$ already guaranteed by δ applies verbatim.
(b) False. Definition 4.2.1 constrains f only at points with $0 < |x-a|$, so the value $f(a)$ is invisible to the limit. Take

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0, \end{cases}$$

for which $\lim_{x\to 0} f(x) = 0$ (any δ works) while $f(0) = 1$.
(c) True, by three applications of Corollary 4.2.4. The constant function -2 has $\lim_{x\to a}(-2) = -2$ straight from Definition 4.2.1 (any δ responds), so part (ii) gives $f(x) - 2 \to L - 2$; part (iii) applied to this function against itself gives $[f(x)-2]^2 \to (L-2)^2$; part (i) with $k = 3$ gives $3[f(x)-2]^2 \to 3(L-2)^2$.
(d) False, because an unbounded g can outrun f. On $A = \mathbf{R} \setminus \{0\}$ (for which $a = 0$ is a limit point) let $f(x) = x$ and $g(x) = 1/x$. Then $\lim_{x\to 0} f(x) = 0$, yet $f(x)g(x) = 1$ for every $x \in A$, so $\lim_{x\to 0} f(x)g(x) = 1 \neq 0$.

Problem 4.2.7. Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Solution. Respond to ϵ with the δ that g supplies for the challenge ϵ/M.
Let $\epsilon > 0$. Since $M > 0$, the number ϵ/M is a legitimate challenge, so Definition 4.2.1 applied to $\lim_{x\to c} g(x) = 0$ furnishes $\delta > 0$ such that $x \in A$ and $0 < |x - c| < \delta$ imply $|g(x) - 0| < \epsilon/M$. For such x,

$$\begin{aligned} |g(x)f(x) - 0| &= |g(x)|\,|f(x)| \\ &\leq |g(x)|\,M \\ &< \frac{\epsilon}{M} \cdot M = \epsilon, \end{aligned}$$

the middle step using $|f(x)| \leq M$, which holds at every point of A.

4.2 Exercises 4.2.8–4.3.3

Problem 4.2.8. Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.
(a) $\lim_{x\to 2} \frac{|x-2|}{x-2}$
(b) $\lim_{x\to 7/4} \frac{|x-2|}{x-2}$
(c) $\lim_{x\to 0} (-1)^{[[1/x]]}$
(d) $\lim_{x\to 0} \sqrt[3]{x}\,(-1)^{[[1/x]]}$
(Here $[[x]]$ denotes the greatest integer less than or equal to x, as in Exercise 4.2.2(c).)

Solution. (a) Does not exist. Put $x_n = 2 + \frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$; both lie in the domain, avoid 2, and converge to 2, while

$$\frac{|x_n - 2|}{x_n - 2} = \frac{1/n}{1/n} = 1, \qquad \frac{|y_n - 2|}{y_n - 2} = \frac{1/n}{-1/n} = -1.$$

Since $1 \neq -1$, the Divergence Criterion (Corollary 4.2.5) applies.
(b) The limit is -1. For every x with $\frac{3}{2} < x < 2$ we have $|x-2| = 2-x$, hence $\frac{|x-2|}{x-2} = -1$ identically on that interval. So given $\epsilon > 0$, take $\delta = \frac{1}{4}$: if $0 < |x - \frac{7}{4}| < \frac{1}{4}$ then $\frac{3}{2} < x < 2$ and

$$\left| \frac{|x-2|}{x-2} - (-1) \right| = 0 < \epsilon.$$

(c) Does not exist. Take $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{2n+1}$, both nonzero with $x_n \to 0$ and $y_n \to 0$. Then $[[1/x_n]] = 2n$ and $[[1/y_n]] = 2n+1$, so

$$(-1)^{[[1/x_n]]} = 1 \to 1, \qquad (-1)^{[[1/y_n]]} = -1 \to -1.$$

Corollary 4.2.5 gives divergence.
(d) The limit is 0. The factor $f(x) = (-1)^{[[1/x]]}$ is bounded by $M = 1$ on $\mathbf{R} \setminus \{0\}$, and $g(x) = \sqrt[3]{x}$ has $\lim_{x\to 0} g(x) = 0$ — given $\epsilon > 0$, take $\delta = \epsilon^3$:

$$0 < |x| < \delta \implies |\sqrt[3]{x} - 0| = |x|^{1/3} < \delta^{1/3} = \epsilon.$$

Exercise 4.2.7 then gives $\lim_{x\to 0} g(x)f(x) = 0$.

Problem 4.2.9. (Infinite Limits). The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge-response style of Definition 4.2.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\epsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:
Definition: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.
(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.
(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$.
(c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.

Solution. (a) Take $\delta = 1/\sqrt{M}$. Given $M > 0$, if $0 < |x - 0| < \delta$ then $x \neq 0$ and $x^2 < \delta^2 = 1/M$, whence

$$\frac{1}{x^2} > M.$$

(b) Definition: for f defined on a domain unbounded above, $\lim_{x\to\infty} f(x) = L$ means that for all $\epsilon > 0$ there exists $M > 0$ such that whenever $x > M$ (and x is in the domain of f), it follows that

$|f(x) - L| < \epsilon$.
For $f(x) = 1/x$ on $(0, \infty)$ and a given $\epsilon > 0$, take $M = 1/\epsilon$:

$$x > M > 0 \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \epsilon.$$

(c) Definition: $\lim_{x\to\infty} f(x) = \infty$ means that for all $M > 0$ there exists $N > 0$ such that whenever $x > N$, it follows that $f(x) > M$. Example: $\lim_{x\to\infty} x^2 = \infty$, since $N = \sqrt{M}$ answers the challenge $M > 0$ — if $x > \sqrt{M}$ then $x^2 > M$.

Problem 4.2.10. (Right and Left Limits). Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."
(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^-} f(x) = M.$$

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Solution. (a) Let $f : A \to \mathbf{R}$. Then $\lim_{x\to a^+} f(x) = L$ means: for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $a < x < a + \delta$ (and $x \in A$), it follows that $|f(x) - L| < \epsilon$. Likewise $\lim_{x\to a^-} f(x) = M$ means: for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $a - \delta < x < a$ (and $x \in A$), it follows that $|f(x) - M| < \epsilon$. (Each statement presumes a is a limit point of $A \cap (a, \infty)$, resp. $A \cap (-\infty, a)$, so it has content.)
(b) Everything rests on the set identity

$$\{x : 0 < |x - a| < \delta\} = (a, a + \delta) \cup (a - \delta, a).$$

($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$ and let $\epsilon > 0$. Definition 4.2.1 supplies $\delta > 0$ with $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. The same δ answers both one-sided challenges, since each of $a < x < a + \delta$ and $a - \delta < x < a$ forces $0 < |x - a| < \delta$.
($\Leftarrow$) Assume both one-sided limits equal L and let $\epsilon > 0$. Pick $\delta_1 > 0$ from the right-hand definition and $\delta_2 > 0$ from the left-hand definition, and set $\delta = \min\{\delta_1, \delta_2\} > 0$. If $0 < |x - a| < \delta$, the identity above puts x in $(a, a + \delta) \subseteq (a, a + \delta_1)$ or in $(a - \delta, a) \subseteq (a - \delta_2, a)$, and in either case

$$|f(x) - L| < \epsilon.$$

Problem 4.2.11. (Squeeze Theorem). Let f, g, and h satisfy $f(x) \le g(x) \le h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well.

Solution. Take $\delta = \min\{\delta_1, \delta_2\}$, where δ_1, δ_2 answer the challenge ϵ for f and for h respectively via Definition 4.2.1 (legitimate, since c is a limit point of A). For $x \in A$ with $0 < |x - c| < \delta$ both estimates hold at once, so

$$L - \epsilon < f(x) \le g(x) \le h(x) < L + \epsilon,$$

that is, $|g(x) - L| < \epsilon$. Hence $\lim_{x\to c} g(x) = L$.
Method (2): Let $(x_n) \subseteq A$ with $x_n \ne c$ and $x_n \to c$. The Sequential Criterion (Theorem 4.2.3) gives $f(x_n) \to L$ and $h(x_n) \to L$, while $f(x_n) \le g(x_n) \le h(x_n)$ for every n; the Squeeze Theorem for sequences (Exercise 2.3.3) yields $g(x_n) \to L$. As (x_n) was arbitrary, Theorem 4.2.3 returns $\lim_{x\to c} g(x) = L$.

Problem 4.3.1. Let $g(x) = \sqrt[3]{x}$.
(a) Prove that g is continuous at $c = 0$.
(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Solution. (a) Take $\delta = \epsilon^3$. If $|x - 0| < \epsilon^3$ then

$$|g(x) - g(0)| = |x|^{1/3} < (\epsilon^3)^{1/3} = \epsilon.$$

(b) Take $\delta = \frac{3}{4}|c|^{2/3}\epsilon$. Writing $a = \sqrt[3]{x}$ and $b = \sqrt[3]{c}$, the suggested identity gives $x - c = (a-b)(a^2 + ab + b^2)$, and the second factor is bounded below away from 0:

$$\begin{aligned} a^2 + ab + b^2 &= \left(a + \tfrac{b}{2}\right)^2 + \tfrac{3}{4}b^2 \\ &\geq \tfrac{3}{4}|c|^{2/3} > 0, \end{aligned}$$

using $c \neq 0$. Hence for every x,

$$\begin{aligned} |g(x) - g(c)| = |a - b| &= \frac{|x-c|}{a^2 + ab + b^2} \\ &\leq \frac{|x-c|}{\frac{3}{4}|c|^{2/3}}, \end{aligned}$$

so $|x - c| < \delta$ forces $|g(x) - g(c)| < \epsilon$.

Problem 4.3.2. To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of $\mathbf{R}$.
(a) Let's say f is *onetinuous* at c if for all $\epsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbf{R}$.
(b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$ we can choose $\delta = \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on $\mathbf{R}$ that is nowhere onetinuous, or explain why there is no such function.
(c) Let's say f is *lesstinuous* at c if for all $\epsilon > 0$ we can choose $0 < \delta < \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on $\mathbf{R}$ that is nowhere equaltinuous, or explain why there is no such function.
(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Solution. (a) Any constant function, say $f(x) = 5$; then $|f(x) - f(c)| = 0 < \epsilon$ for every x, so $\delta = 1$ always works. (These are the only examples: onetinuity at c says $|f(x) - f(c)| < \epsilon$ for all $\epsilon > 0$ whenever $|x - c| < 1$, i.e. $f \equiv f(c)$ on $(c-1, c+1)$, and chaining such unit intervals across $\mathbf{R}$ makes f constant.)
(b) $f(x) = x$. Unwinding the definition, f is equaltinuous at c exactly when $|f(x) - f(c)| \leq |x - c|$ for all x: letting $\epsilon \downarrow |x - c|$ in the hypothesis gives the inequality, and the inequality clearly returns the hypothesis. The identity function satisfies it with equality, hence is equaltinuous on $\mathbf{R}$; and it is constant on no interval, so by the parenthetical in (a) it is nowhere onetinuous.
(c) $f(x) = 2x$. Given $\epsilon > 0$ choose $\delta = \epsilon/2$, which satisfies $0 < \delta < \epsilon$, and $|x - c| < \epsilon/2$ gives $|f(x) - f(c)| = 2|x - c| < \epsilon$. It is nowhere equaltinuous because $|f(x) - f(c)| = 2|x - c| > |x - c|$ for every $x \neq c$, violating the criterion in (b).
(d) Yes to both, so the two notions coincide. A lesstinuous f produces, for each ϵ, a $\delta > 0$ meeting the requirement of Definition 4.3.1 — the extra demand $\delta < \epsilon$ is simply ignored. Conversely, if f is continuous at c and $\epsilon > 0$ is given, let δ be as in Definition 4.3.1 and replace it by

$$\delta' = \min\{\delta,\ \epsilon/2\},$$

which satisfies $0 < \delta' < \epsilon$; since $\delta' \leq \delta$, the implication $|x-c| < \delta' \Rightarrow |f(x) - f(c)| < \epsilon$ still holds.

Problem 4.3.3. (a) Supply a proof for Theorem 4.3.9 using the ϵ–δ characterization of continuity.
(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).
(Theorem 4.3.9, Composition of Continuous Functions: given $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.)

Solution. (a) Let $\epsilon > 0$. Continuity of g at $f(c) \in B$ supplies $\gamma > 0$ such that

$$y \in B,\ |y - f(c)| < \gamma \implies |g(y) - g(f(c))| < \epsilon,$$

and continuity of f at c, applied with this γ in the role of the challenge, supplies $\delta > 0$ such that

$$x \in A,\ |x - c| < \delta \implies |f(x) - f(c)| < \gamma.$$

Because $f(A) \subseteq B$, the value $y = f(x)$ is a legitimate input for the first implication, and the two combine: for $x \in A$ with $|x - c| < \delta$,

$$|g(f(x)) - g(f(c))| < \epsilon.$$

(b) Let $(x_n) \to c$ with $(x_n) \subseteq A$. Theorem 4.3.2 (iii) applied to f at c gives $f(x_n) \to f(c)$. The terms $f(x_n)$ lie in B and $f(c) \in B$, so Theorem 4.3.2 (iii) applied to g at the point $f(c)$ gives

$$g(f(x_n)) \to g(f(c)).$$

Since $(x_n) \to c$ in A was arbitrary, the same theorem read in the other direction shows $g \circ f$ is continuous at c.

4.3 Exercises 4.3.4–4.3.10

Problem 4.3.4. Assume f and g are defined on all of $\mathbf{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.
(a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.
(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution. (a) Take $p = q = r = 0$ with

$$f(x) = 0 \quad \text{for all } x, \qquad g(y) = \begin{cases} 0 & \text{if } y \neq 0 \\ 1 & \text{if } y = 0. \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0 = q$ and $\lim_{y \to 0} g(y) = 0 = r$ (the value $g(0)$ is irrelevant to the limit, by Definition 4.2.1). But $g(f(x)) = g(0) = 1$ for every x, so $\lim_{x \to 0} g(f(x)) = 1 \neq r$.
(b) Continuity of f at p and of g at q turns the two hypotheses into $f(p) = q$ and $g(q) = r$. Theorem 4.3.9 makes $g \circ f$ continuous at p, so

$$\lim_{x \to p} g(f(x)) = g(f(p)) = g(q) = r.$$

(c) Assuming only f continuous does not suffice: the f of part (a) is constant, hence continuous, and the conclusion fails there.

Assuming only g continuous does suffice. Then $g(q) = r$. Let $(x_n) \to p$ with $x_n \neq p$; the sequential criterion for functional limits (Theorem 4.2.3) gives $f(x_n) \to q$, and Theorem 4.3.2 (iii) for g at q gives

$$g(f(x_n)) \to g(q) = r.$$

As this holds for every such sequence, Theorem 4.2.3 returns $\lim_{x\to p} g(f(x)) = r$.

Problem 4.3.5. Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continuous at c.

Solution. Take the δ that isolates c. By definition of isolated point there is a $\delta > 0$ with

$$V_\delta(c) \cap A = \{c\},$$

and this one δ answers every $\epsilon > 0$: if $x \in A$ satisfies $|x - c| < \delta$ then $x = c$, whence

$$|f(x) - f(c)| = 0 < \epsilon.$$

Problem 4.3.6. Provide an example of each or explain why the request is impossible.
(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0.
(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)+g(x)$ is continuous at 0.
(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.
(d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.
(e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

Solution. (a) Let f be Dirichlet's function and $g = 1 - f$:

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases} \qquad g(x) = \begin{cases} 0 & x \in \mathbf{Q} \\ 1 & x \notin \mathbf{Q}. \end{cases}$$

Neither is continuous at 0 — f by Exercise 4.3.7 (a), and g because otherwise $f = 1 - g$ would be by Theorem 4.3.4 (i) and (ii). But $f + g \equiv 1$ and $fg \equiv 0$, since one factor vanishes at each x; constants are continuous.
(b) Impossible. If f and $f + g$ were both continuous at 0, then by Theorem 4.3.4 (i) and (ii) so is

$$g = (f + g) + (-1)f,$$

contradicting the hypothesis on g.
(c) Take $f \equiv 0$ and g Dirichlet's function. Then $fg \equiv 0$ is continuous at 0 while g is not.
(d) Take

$$f(x) = \begin{cases} 2 & x \in \mathbf{Q} \\ 1/2 & x \notin \mathbf{Q}, \end{cases}$$

whose values are the reciprocal pair $\{2, 1/2\}$, so that

$$f(x) + \frac{1}{f(x)} = 2 + \tfrac{1}{2} = \tfrac{5}{2}$$

for every x — a constant, hence continuous. That f itself is discontinuous at 0 follows as in Exercise 4.3.7 (a): rational and irrational sequences tending to 0 give limits 2 and 1/2.
(e) Impossible. The cube root $\psi(y) = \sqrt[3]{y}$ is continuous on $\mathbf{R}$ (Exercise 4.3.1), so if $h = f^3$ were continuous at 0, Theorem 4.3.9 applied to $\psi \circ h$ would make

$$\psi(h(x)) = \sqrt[3]{[f(x)]^3} = f(x)$$

continuous at 0.

Problem 4.3.7. (a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbf{R}$.
(b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.
(c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbf{R}$. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$.)
(Here Dirichlet's function is $g(x) = 1$ for $x \in \mathbf{Q}$ and $g(x) = 0$ for $x \notin \mathbf{Q}$; Thomae's function is $t(x) = 1$ if $x = 0$, $t(x) = 1/n$ if $x = m/n \in \mathbf{Q} \setminus \{0\}$ is in lowest terms with $n > 0$, and $t(x) = 0$ if $x \notin \mathbf{Q}$.)

Solution. (a) Fix $c \in \mathbf{R}$. Because $\mathbf{Q}$ and $\mathbf{I}$ are both dense in $\mathbf{R}$ (Theorem 1.4.3 and Corollary 1.4.4), we may choose $x_n \in \mathbf{Q} \cap (c - 1/n,\, c + 1/n)$ and $y_n \in \mathbf{I} \cap (c - 1/n,\, c + 1/n)$, so that $|x_n - c| < 1/n$ and $|y_n - c| < 1/n$, whence $(x_n) \to c$ and $(y_n) \to c$. Then

$$g(x_n) = 1 \to 1, \qquad g(y_n) = 0 \to 0.$$

Since $g(c)$ cannot equal both 1 and 0, one of these sequences has g-image failing to converge to $g(c)$, and Corollary 4.3.3 declares g discontinuous at c.
(b) Let $c \in \mathbf{Q}$, so $t(c) > 0$ (it is 1 if $c = 0$ and $1/n > 0$ otherwise). Picking $y_n \in \mathbf{I}$ with $(y_n) \to c$ exactly as in (a),

$$t(y_n) = 0 \to 0 \neq t(c),$$

so Corollary 4.3.3 gives discontinuity at every rational point.
(c) Let $c \in \mathbf{I}$, so $t(c) = 0$, and let $(x_n) \to c$; by Theorem 4.3.2 (iii) it suffices to prove $t(x_n) \to 0$. Fix $\epsilon > 0$ and set

$$A_\epsilon = \{x \in \mathbf{R} : t(x) \geq \epsilon\}.$$

Every $x \in A_\epsilon$ is rational, and is either 0 or of the form m/n in lowest terms with $1/n \geq \epsilon$, i.e. $1 \leq n \leq 1/\epsilon$. Consequently

$$F = A_\epsilon \cap [c - 1,\, c + 1]$$

is finite: only finitely many denominators $n \leq 1/\epsilon$ are available, and for each the interval $[c-1, c+1]$ of length 2 contains at most $2n + 1$ fractions m/n.
Since $c \notin F$ (as $t(c) = 0 < \epsilon$), the number

$$\delta = \min\bigl(\{1\} \cup \{|y - c| : y \in F\}\bigr)$$

is the minimum of a finite set of strictly positive numbers, hence $\delta > 0$. Choose N with $|x_n - c| < \delta$ for all $n \geq N$. Such an x_n lies in $[c - 1, c + 1]$ and is not a point of F, so $x_n \notin A_\epsilon$; that is,

$$|t(x_n) - t(c)| = t(x_n) < \epsilon \qquad \text{for all } n \geq N.$$

Thus $t(x_n) \to t(c)$, and t is continuous at c.

Problem 4.3.8. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbf{R}$.
(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.
(b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.
(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution. All three are true.
(a) Take $x_n = 1 - 1/n$. Then $(x_n) \to 1$ with $x_n < 1$, so $g(x_n) \geq 0$ for every n, and Theorem 4.3.2 (iii)

together with the Order Limit Theorem 2.3.4 gives

$$g(1) \;=\; \lim_{n\to\infty} g(x_n) \;\geq\; 0.$$

(b) Fix $x \in \mathbf{R}$. By the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) choose $r_n \in \mathbf{Q}$ with $x < r_n < x + 1/n$, so that $(r_n) \to x$. Since g is continuous at x, Theorem 4.3.2 (iii) yields

$$g(x) \;=\; \lim_{n\to\infty} g(r_n) \;=\; \lim_{n\to\infty} 0 \;=\; 0.$$

(c) Apply continuity at x_0 with the particular choice $\epsilon = g(x_0) > 0$: there is a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < g(x_0) \implies g(x) > 0.$$

Thus g is strictly positive on the whole interval $V_\delta(x_0) = (x_0 - \delta,\, x_0 + \delta)$, and $(x_0 - \delta, x_0 + \delta) \sim \mathbf{R}$ by Exercise 1.5.4 (a), which is uncountable by Theorem 1.5.6 (ii).

Problem 4.3.9. Assume $h : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution. K contains its limit points, which is Definition 3.2.7. If x is a limit point of K, Theorem 3.2.5 supplies a sequence $(x_n) \subseteq K$ with $(x_n) \to x$, and continuity of h at x gives, via Theorem 4.3.2 (iii),

$$h(x) \;=\; \lim_{n\to\infty} h(x_n) \;=\; \lim_{n\to\infty} 0 \;=\; 0,$$

so $x \in K$.

Problem 4.3.10. Observe that if a and b are real numbers, then

$$\max\{a, b\} \;=\; \tfrac{1}{2}\big[(a+b) + |a-b|\big].$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) \;=\; \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.
(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on $\mathbf{R}$ by

$$f_n(x) \;=\; \begin{cases} 1 & \text{if } |x| \geq 1/n, \\ n|x| & \text{if } |x| < 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

Solution. (a) Induct on n; the case $n = 1$ is trivial. For $n = 2$ the displayed identity gives

$$\max\{f_1, f_2\} = \tfrac{1}{2}(f_1 + f_2) + \tfrac{1}{2}\big|f_1 - f_2\big|,$$

and every piece is continuous: the sums and scalar multiples by Theorem 4.3.4 (i), (ii), and $|f_1 - f_2| = a \circ (f_1 - f_2)$ by Theorem 4.3.9, where $a(t) = |t|$ is continuous on all of $\mathbf{R}$ because $\big||t| - |c|\big| \leq |t - c|$ makes $\delta = \epsilon$ work. The inductive step is

$$\max\{f_1, \ldots, f_{n+1}\} = \max\big\{\max\{f_1, \ldots, f_n\},\, f_{n+1}\big\}.$$

(b) The supremum is

$$h(x) \;=\; \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Indeed $f_n(0) = n \cdot 0 = 0$ for every n, so $h(0) = 0$. If $x \neq 0$, the Archimedean Property (Theorem 1.4.2) supplies $N \in \mathbf{N}$ with $1/N < |x|$, whence $f_N(x) = 1$; and every f_n satisfies $f_n(x) \leq 1$ (either $f_n(x) = 1$, or $f_n(x) = n|x| < 1$). Thus $\sup_n f_n(x) = 1$.
So the extension fails: each f_n is continuous, but taking $x_n = 1/n \to 0$ gives $h(x_n) = 1 \not\to 0 = h(0)$, and h is discontinuous at 0 by Corollary 4.3.3.

4.4 Exercises 4.3.11–4.4.3

Problem 4.3.11. (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbf{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c\,|x - y|$$

for all $x, y \in \mathbf{R}$.
(a) Show that f is continuous on $\mathbf{R}$.
(b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$\big(y_1,\ f(y_1),\ f(f(y_1)),\ \ldots\big).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.
(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.
(d) Finally, prove that if x is any arbitrary point in $\mathbf{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

Solution. (a) Given $\epsilon > 0$ and $p \in \mathbf{R}$, take $\delta = \epsilon$. Then $|x - p| < \delta$ implies

$$|f(x) - f(p)| \ \leq\ c\,|x - p| \ <\ c\epsilon \ <\ \epsilon,$$

since $c < 1$. This is Definition 4.3.1, so f is continuous at every p.
(b) Iterating the contraction bound along the sequence gives

$$|y_{n+1} - y_n| \ =\ |f(y_n) - f(y_{n-1})| \ \leq\ c\,|y_n - y_{n-1}| \ \leq\ \cdots \ \leq\ c^{n-1}|y_2 - y_1|.$$

Hence for $m > n$, by the triangle inequality and the geometric series,

$$\begin{aligned} |y_m - y_n| &\leq \sum_{k=n}^{m-1} |y_{k+1} - y_k| \ \leq\ |y_2 - y_1| \sum_{k=n}^{m-1} c^{k-1} \\ &\leq\ |y_2 - y_1|\,\frac{c^{n-1}}{1 - c}. \end{aligned}$$

Since $0 < c < 1$ we have $c^{n-1} \to 0$ (Example 2.5.3), so given $\epsilon > 0$ we may pick N with $|y_2 - y_1|c^{N-1}/(1 - c) < \epsilon$, and then $|y_m - y_n| < \epsilon$ for all $m > n \geq N$. Thus (y_n) is Cauchy and converges by the Cauchy Criterion (Theorem 2.6.4); set $y = \lim y_n$.
(c) f is continuous at y and $(y_n) \to y$, so $f(y_n) \to f(y)$ by Theorem 4.3.2 (iii). But $f(y_n) = y_{n+1}$, and (y_{n+1}) is a subsequence of (y_n), hence also converges to y (Theorem 2.5.2). Uniqueness of limits (Theorem 2.2.7) forces

$$f(y) \ =\ y.$$

If z also satisfies $f(z) = z$, then

$$|y - z| \ =\ |f(y) - f(z)| \ \leq\ c\,|y - z| \quad\Longrightarrow\quad (1 - c)\,|y - z| \leq 0,$$

and since $1 - c > 0$ this gives $|y - z| = 0$, i.e. $z = y$.
(d) Let $x_1 = x$ and $x_{n+1} = f(x_n)$. Using $f(y) = y$ and iterating,

$$|x_n - y| \ =\ |f(x_{n-1}) - f(y)| \ \leq\ c\,|x_{n-1} - y| \ \leq\ \cdots \ \leq\ c^{n-1}|x - y|.$$

Because $c^{n-1} \to 0$, the Squeeze Theorem (Exercise 2.3.3) gives $|x_n - y| \to 0$, i.e. $(x_n) \to y$.

Problem 4.3.12. Let $F \subseteq \mathbf{R}$ be a nonempty closed set and define $g(x) = \inf\{|x-a| : a \in F\}$. Show that g is continuous on all of $\mathbf{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Solution. g is 1-Lipschitz, so $\delta = \epsilon$ proves continuity at every point.
(Each infimum exists: $\{|x-a| : a \in F\}$ is nonempty since $F \neq \emptyset$ and is bounded below by 0, so the greatest-lower-bound form of the Axiom of Completeness — Exercise 1.3.3 — applies.)
Fix $x, y \in \mathbf{R}$. For every $a \in F$ the triangle inequality gives

$$g(x) \;\leq\; |x-a| \;\leq\; |x-y| + |y-a|,$$

so $g(x) - |x-y|$ is a lower bound for $\{|y-a| : a \in F\}$, and therefore

$$g(x) - |x-y| \;\leq\; g(y), \qquad \text{i.e.} \qquad g(x) - g(y) \;\leq\; |x-y|.$$

Interchanging x and y gives $g(y) - g(x) \leq |x-y|$, hence

$$|g(x) - g(y)| \;\leq\; |x-y|.$$

Given $\epsilon > 0$, choose $\delta = \epsilon$; then $|x-y| < \delta$ implies $|g(x)-g(y)| < \epsilon$, which is Definition 4.3.1.
Now suppose $x \notin F$ but $g(x) = 0$. For each $n \in \mathbf{N}$ the number $1/n$ is then not a lower bound for $\{|x-a| : a \in F\}$ (Exercise 1.3.1 (b)), so there exists $a_n \in F$ with

$$0 \;\leq\; |x - a_n| \;<\; 1/n.$$

Thus $(a_n) \to x$, and $a_n \neq x$ for every n because $x \notin F$. By Theorem 3.2.5 this makes x a limit point of F, and F is closed, so $x \in F$ by Definition 3.2.7 — a contradiction. Hence $g(x) > 0$, and in particular $g(x) \neq 0$, for all $x \notin F$.

Problem 4.3.13. Let f be a function defined on all of $\mathbf{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.
(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.
(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbf{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbf{Z}$. Now, prove that $f(r) = kr$ for any rational number r.
(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbf{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbf{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution. (a) Setting $x = y = 0$ in the additive condition gives $f(0) = f(0) + f(0)$, so $f(0) = 0$. Then for any x,

$$0 \;=\; f(0) \;=\; f\big(x + (-x)\big) \;=\; f(x) + f(-x),$$

hence $f(-x) = -f(x)$.
(b) $f(1) = k \cdot 1$, and if $f(n) = kn$ then

$$f(n+1) \;=\; f(n) + f(1) \;=\; kn + k \;=\; k(n+1),$$

so $f(n) = kn$ for all $n \in \mathbf{N}$ by induction. For $z \in \mathbf{Z}$: the case $z \in \mathbf{N}$ is done, $f(0) = 0 = k \cdot 0$, and for $z = -n$ with $n \in \mathbf{N}$ part (a) gives $f(-n) = -f(n) = -kn = k(-n)$.
For $r = p/q$ with $p \in \mathbf{Z}$ and $q \in \mathbf{N}$, additivity applied to q copies of p/q gives

$$\begin{aligned} kp = f(p) &= f\left(\tfrac{p}{q} + \cdots + \tfrac{p}{q}\right) \\ &= q\, f\left(\tfrac{p}{q}\right), \end{aligned}$$

so $f(r) = f(p/q) = kp/q = kr$.

(c) Fix $c \in \mathbf{R}$ and let $\epsilon > 0$. Continuity at 0 (with $f(0) = 0$) supplies $\delta > 0$ such that $|t| < \delta$ implies $|f(t)| < \epsilon$. Additivity gives $f(x) - f(c) = f(x - c)$, so

$$|x - c| < \delta \implies |f(x) - f(c)| = |f(x - c)| < \epsilon.$$

The same δ works at every c, so f is continuous on $\mathbf{R}$.
Now set $h(x) = f(x) - kx$. It is continuous on $\mathbf{R}$ by Theorem 4.3.4 (i), (ii) and Example 4.3.5, and $h(r) = 0$ for every $r \in \mathbf{Q}$ by part (b). By Exercise 4.3.8 (b), $h \equiv 0$; that is,

$$f(x) \;=\; kx \qquad \text{for all } x \in \mathbf{R}.$$

Problem 4.3.14. (a) Let F be a closed set. Construct a function $f : \mathbf{R} \to \mathbf{R}$ such that the set of points where f fails to be continuous is precisely F. (The concept of the interior of a set, discussed in Exercise 3.2.14, may be useful.)
(b) Now consider an open set O. Construct a function $g : \mathbf{R} \to \mathbf{R}$ whose set of discontinuous points is precisely O. (For this problem, the function in Exercise 4.3.12 may be useful.)

Solution. (a) Take the signed Dirichlet function supported on F:

$$f(x) \;=\; \begin{cases} 1 & \text{if } x \in F \cap \mathbf{Q}, \\ -1 & \text{if } x \in F \setminus \mathbf{Q}, \\ 0 & \text{if } x \notin F. \end{cases}$$

Write F° for the interior (Exercise 3.2.14) and check the three cases.
(i) $x \notin F$. Since F is closed, F^c is open (Theorem 3.2.13), so there is a $V_\delta(x) \subseteq F^c$ on which f is identically 0. Then $|f(y) - f(x)| = 0 < \epsilon$ whenever $|y - x| < \delta$, for any $\epsilon > 0$: f is continuous at x.
(ii) $x \in F^\circ$. Choose $\delta > 0$ with $V_\delta(x) \subseteq F$. For each n, Theorem 1.4.3 and Corollary 1.4.4 give a rational r_n and an irrational s_n inside the interval $\big(x,\, x + \min\{\delta, 1/n\}\big)$; both lie in F, so

$$(r_n) \to x \;\text{ with } f(r_n) = 1, \qquad (s_n) \to x \;\text{ with } f(s_n) = -1.$$

Since $f(x)$ equals 1 or -1, one of these two sequences has $f(\cdot\,) \not\to f(x)$, and Corollary 4.3.3 makes f discontinuous at x.
(iii) $x \in F \setminus F^\circ$. For each n the neighborhood $V_{1/n}(x)$ is not contained in F (otherwise x would be interior), so pick $t_n \in V_{1/n}(x)$ with $t_n \notin F$. Then $(t_n) \to x$ and $f(t_n) = 0$, while $|f(x)| = 1$; Corollary 4.3.3 again gives discontinuity.
So f is discontinuous exactly on $F^\circ \cup (F \setminus F^\circ) = F$.
(b) If $O = \mathbf{R}$, Dirichlet's function from Section 4.1 is discontinuous at every point and we are done; so assume $O \neq \mathbf{R}$. Then $F = O^c$ is a nonempty closed set (Theorem 3.2.13), and Exercise 4.3.12 makes

$$d(x) \;=\; \inf\{|x - a| : a \in F\}$$

continuous on $\mathbf{R}$, with $d(x) > 0$ exactly when $x \notin F$, i.e. exactly when $x \in O$ (for $x \in F$ the choice $a = x$ forces $d(x) = 0$). Define

$$g(x) \;=\; \begin{cases} d(x) & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Discontinuity on O: let $x \in O$, so $d(x) > 0$. Choose rationals $(r_n) \to x$ and irrationals $(s_n) \to x$ (Theorem 1.4.3, Corollary 1.4.4). Then

$$g(r_n) = d(r_n) \to d(x) > 0, \qquad g(s_n) = 0 \to 0,$$

the first limit by continuity of d and Theorem 4.3.2 (iii). The two limits differ, so at least one disagrees with $g(x)$, and g is discontinuous at x by Corollary 4.3.3.

Continuity off O: let $x \notin O$, so $d(x) = 0$ and hence $g(x) = 0$ in either branch. Given $\epsilon > 0$, continuity of d at x supplies $\delta > 0$ with $d(y) = |d(y) - d(x)| < \epsilon$ whenever $|y - x| < \delta$. Since $0 \le g(y) \le d(y)$ for every y,

$$|y - x| < \delta \implies |g(y) - g(x)| = g(y) \le d(y) < \epsilon.$$

Thus the set of discontinuities of g is precisely O.

Problem 4.4.1. (a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.
(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbf{R}$.
(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Solution. (a) The identity $g(x) = x$ is continuous on $\mathbf{R}$ (given $\epsilon > 0$, take $\delta = \epsilon$), and by the Algebraic Continuity Theorem (Theorem 4.3.4 (iii)) products of continuous functions are continuous; hence $f = g \cdot g \cdot g$ is continuous on $\mathbf{R}$.
(b) Take $\epsilon_0 = 3$ and the sequences

$$x_n = n + \frac{1}{n}, \qquad y_n = n.$$

Then $|x_n - y_n| = 1/n \to 0$, while

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left(n + \tfrac{1}{n}\right)^3 - n^3 \\ &= 3n^2 \cdot \tfrac{1}{n} + 3n \cdot \tfrac{1}{n^2} + \tfrac{1}{n^3} \\ &= 3n + \tfrac{3}{n} + \tfrac{1}{n^3} \ \ge\ 3 = \epsilon_0 \end{aligned}$$

for every $n \in \mathbf{N}$. By Theorem 4.4.5, f fails to be uniformly continuous on $\mathbf{R}$.
(c) Let $A \subseteq \mathbf{R}$ be bounded, say $A \subseteq [-M, M]$ with $M > 0$. The interval $[-M, M]$ is closed and bounded, hence compact (Theorem 3.3.4), and f is continuous on it by part (a); so f is uniformly continuous on $[-M, M]$ by Theorem 4.4.7. A δ that works for all pairs in $[-M, M]$ works in particular for all pairs in A, so f is uniformly continuous on A.
Method (2): for $x, y \in A$,

$$|x^3 - y^3| = |x - y|\,|x^2 + xy + y^2| \le 3M^2 |x - y|,$$

so $\delta = \epsilon/(3M^2)$ answers any $\epsilon > 0$.

Problem 4.4.2. (a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?
(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?
(c) Is $h(x) = x \sin(1/x)$ uniformly continuous on $(0, 1)$?

Solution. (a) No. Take $\epsilon_0 = 1$ and

$$x_n = \frac{1}{n}, \qquad y_n = \frac{1}{2n} \qquad (n \ge 2),$$

both in $(0, 1)$. Then $|x_n - y_n| = 1/(2n) \to 0$ while

$$|f(x_n) - f(y_n)| = |n - 2n| = n \ge 1 = \epsilon_0,$$

so Theorem 4.4.5 shows f is not uniformly continuous on $(0, 1)$.
(b) Yes — g is Lipschitz with constant 1. For $x, y \in (0, 1)$, rationalizing gives

$$\begin{aligned} |g(x) - g(y)| &= \frac{|x^2 - y^2|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}} \\ &\le \frac{|x - y|\,|x + y|}{2} \ \le\ |x - y|, \end{aligned}$$

using $\sqrt{x^2+1} \geq 1$ in the denominator and $|x+y| < 2$ in the numerator. Given $\epsilon > 0$, take $\delta = \epsilon$.
(c) Yes. Define $\tilde{h}$ on $[0,1]$ by $\tilde{h}(x) = x\sin(1/x)$ for $x \neq 0$ and $\tilde{h}(0) = 0$. On $(0,1]$ the function is continuous by the Algebraic Continuity Theorem (Theorem 4.3.4) and Theorem 4.3.9, and at 0 the bound

$$|\tilde{h}(x) - \tilde{h}(0)| = |x|\,|\sin(1/x)| \leq |x|$$

shows $\delta = \epsilon$ works. So $\tilde{h}$ is continuous on the compact set $[0,1]$ (Theorem 3.3.4), hence uniformly continuous there by Theorem 4.4.7. Restricting to $(0,1)$, where $\tilde{h} = h$, preserves the δ.

Problem 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1,\infty)$ but not on the set $(0,1]$.

Solution. On $[1,\infty)$ the function is Lipschitz with constant 2: for $x, y \geq 1$,

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{y^2}\right| &= \frac{|y-x|\,(x+y)}{x^2y^2} = |x-y|\left(\frac{1}{xy^2} + \frac{1}{x^2y}\right)\\ &\leq 2|x-y|,\end{aligned}$$

since $x, y \geq 1$ makes each of $1/(xy^2)$ and $1/(x^2y)$ at most 1. Given $\epsilon > 0$, choose $\delta = \epsilon/2$; then $|x-y| < \delta$ forces $|f(x) - f(y)| < \epsilon$.
On $(0,1]$ take $\epsilon_0 = 1$ and

$$x_n = \frac{1}{n}, \qquad y_n = \frac{1}{n+1}.$$

Then $|x_n - y_n| = \dfrac{1}{n(n+1)} \to 0$, while

$$|f(x_n) - f(y_n)| = (n+1)^2 - n^2 = 2n+1 \geq 1 = \epsilon_0$$

for all $n \in \mathbf{N}$. By Theorem 4.4.5, f is not uniformly continuous on $(0,1]$.

4.5 Exercises 4.4.4–4.4.10

Problem 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.
(a) If f is continuous on $[a,b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a,b]$ (meaning $1/f$ has bounded range).
(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.
(c) If f is defined on $\mathbf{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbf{R}$.

Solution. (a) True. The set $[a,b]$ is compact (Theorem 3.3.4), so by the Extreme Value Theorem (Theorem 4.4.2) f attains a minimum value $m = f(x_0)$ for some $x_0 \in [a,b]$, and $m > 0$ by hypothesis. Hence

$$0 < \frac{1}{f(x)} \leq \frac{1}{m} \qquad \text{for all } x \in [a,b].$$

(b) True. Suppose $f(A)$ were unbounded, and pick $x_n \in A$ with $|f(x_n)| \geq n$. Since A is bounded, the Bolzano–Weierstrass Theorem (Theorem 2.5.5) supplies a convergent subsequence (x_{n_k}), which is therefore Cauchy. Uniformly continuous functions preserve Cauchy sequences (Exercise 4.4.13 (a)), so $\big(f(x_{n_k})\big)$ is Cauchy and hence bounded (Lemma 2.6.3). This contradicts $|f(x_{n_k})| \geq n_k \to \infty$.
(c) False. Define

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

For any compact $K \subseteq \mathbf{R}$ the image $f(K)$ is one of $\{0\}$, $\{1\}$, $\{0,1\}$; a finite set is closed and bounded, hence compact (Theorem 3.3.4). But f is discontinuous at 0: with $x_n = 1/n \to 0$ we get $f(x_n) = 0 \to$

$0 \neq 1 = f(0)$, so continuity fails by the sequential criterion (Corollary 4.3.3).

Problem 4.4.5. Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Solution. Take $\delta = \min\{\delta_1, \delta_2\}$, where δ_1 and δ_2 answer $\epsilon/2$ on $(a, b]$ and on $[b, c)$ respectively; the point b lies in both pieces, and that is what lets the two estimates be chained.
Let $\epsilon > 0$ and fix such a δ. Let $x, y \in (a, c)$ with $|x - y| < \delta$, and assume without loss of generality $x \leq y$.
(i) Both $x, y \leq b$. Then $x, y \in (a, b]$ and $|g(x) - g(y)| < \epsilon/2 < \epsilon$.
(ii) Both $x, y \geq b$. Then $x, y \in [b, c)$ and $|g(x) - g(y)| < \epsilon/2 < \epsilon$.
(iii) $x < b < y$. Then $x, b \in (a, b]$ with $|x - b| < |x - y| < \delta_1$, and $b, y \in [b, c)$ with $|b - y| < |x - y| < \delta_2$, so

$$\begin{aligned} |g(x) - g(y)| &\leq |g(x) - g(b)| + |g(b) - g(y)| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

In every case $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$, which is Definition 4.4.4 on (a, c).

Problem 4.4.6. Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.
(a) A continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;
(b) A uniformly continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;
(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution. (a) Take $f(x) = 1/x$ on $(0, 1)$ and $x_n = 1/n$. The sequence (x_n) converges to 0, hence is Cauchy (Theorem 2.6.2), and f is continuous on $(0, 1)$ by the Algebraic Continuity Theorem (Theorem 4.3.4 (iv)), the denominator being nonzero there. But $f(x_n) = n$ is unbounded, so it is not Cauchy (Lemma 2.6.3).
(b) Impossible: uniformly continuous functions preserve Cauchy sequences. Given $\epsilon > 0$, choose δ as in Definition 4.4.4 and then N with $|x_n - x_m| < \delta$ for all $m, n \geq N$; for such m, n,

$$|f(x_n) - f(x_m)| < \epsilon.$$

(This is Exercise 4.4.13 (a).)
(c) Impossible. A Cauchy sequence $(x_n) \subseteq [0, \infty)$ converges to some $x \in \mathbf{R}$ by the Cauchy Criterion (Theorem 2.6.4), and $x \geq 0$ by the Order Limit Theorem (Theorem 2.3.4), so x lies in the domain $[0, \infty)$. Continuity of f at x then gives $f(x_n) \to f(x)$ (Theorem 4.3.2 (iii)), and convergent sequences are Cauchy (Theorem 2.6.2).

Problem 4.4.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. Given $\epsilon > 0$, take $\delta = \epsilon^2$; the work is the inequality

$$|\sqrt{x} - \sqrt{y}\,| \leq \sqrt{|x - y|} \qquad (x, y \geq 0).$$

To see it, assume without loss of generality $x \geq y \geq 0$. Then

$$\begin{aligned} \left(\sqrt{y} + \sqrt{x - y}\,\right)^2 &= y + 2\sqrt{y}\sqrt{x - y} + (x - y) \\ &\geq x = (\sqrt{x}\,)^2, \end{aligned}$$

and since both bases are nonnegative, $\sqrt{y} + \sqrt{x - y} \geq \sqrt{x}$, i.e. $\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}$.

Now if $x, y \in [0, \infty)$ satisfy $|x - y| < \delta = \epsilon^2$, then

$$\begin{aligned} |f(x) - f(y)| &= |\sqrt{x} - \sqrt{y}\,| \\ &\le \sqrt{|x - y|} < \sqrt{\epsilon^2} = \epsilon, \end{aligned}$$

the last step using that $t \mapsto \sqrt{t}$ is increasing on $[0, \infty)$. Since δ depends only on ϵ, Definition 4.4.4 is satisfied.

Problem 4.4.8. Give an example of each of the following, or provide a short argument for why the request is impossible.
(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.
(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.
(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Solution. (a) Impossible. $[0, 1]$ is closed and bounded, hence compact (Theorem 3.3.4), so $f([0, 1])$ is compact by Theorem 4.4.1; but $(0, 1)$ is not closed and therefore not compact. (Equivalently: the Extreme Value Theorem 4.4.2 forces f to attain $\sup f([0, 1])$, which $(0, 1)$ never contains.)
(b) Possible. Take

$$f(x) = \tfrac{1}{2}\big(1 + \sin(2\pi x)\big), \qquad x \in (0, 1).$$

Then f is continuous with $f(x) \in [0, 1]$, and $f(1/4) = 1$, $f(3/4) = 0$; the Intermediate Value Theorem (Theorem 4.5.1) applied on $[1/4, 3/4]$ supplies every value in between. Hence the range is exactly $[0, 1]$.
(c) Possible. Take

$$f(x) = \tfrac{1}{2}\Big(1 + (1 - x)\sin(1/x)\Big), \qquad x \in (0, 1].$$

This is continuous on $(0, 1]$ (Theorems 4.3.4 and 4.3.9), and

$$\begin{aligned} \Big|f(x) - \tfrac{1}{2}\Big| &= \tfrac{1}{2}(1 - x)\,|\sin(1/x)| \\ &\le \tfrac{1-x}{2} \ < \ \tfrac{1}{2}, \end{aligned}$$

so $f(x) \in (0, 1)$ for every $x \in (0, 1]$: the range is contained in $(0, 1)$ and never reaches the endpoints. For surjectivity put

$$x_n = \frac{1}{\pi/2 + 2n\pi}, \qquad y_n = \frac{1}{3\pi/2 + 2n\pi},$$

so that $y_n < x_n$ and

$$f(x_n) = 1 - \tfrac{x_n}{2} \to 1, \qquad f(y_n) = \tfrac{y_n}{2} \to 0.$$

Given $t \in (0, 1)$, choose n with $f(y_n) < t < f(x_n)$ and apply the Intermediate Value Theorem on $[y_n, x_n] \subseteq (0, 1]$ to get c with $f(c) = t$. Hence the range is exactly $(0, 1)$.

Problem 4.4.9. (Lipschitz Functions). A function $f : A \to \mathbf{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le M$$

for all $x \ne y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.
(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.
(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. (a) Given $\epsilon > 0$, take $\delta = \epsilon/M$. The Lipschitz bound is exactly the statement

$$|f(x) - f(y)| \le M\,|x - y| \qquad \text{for all } x, y \in A$$

(the case $x = y$ being trivial), so $|x - y| < \delta$ gives $|f(x) - f(y)| \le M|x - y| < M\delta = \epsilon$. Since δ depends

only on ϵ, Definition 4.4.4 is satisfied.
(b) No. Take $f(x) = \sqrt{x}$ on $A = [0, 1]$.
Uniformly continuous. f is continuous on $[0, 1]$ (Theorem 4.3.2 (iii) together with Exercise 2.3.1), and $[0, 1]$ is compact (Theorem 3.3.4), so Theorem 4.4.7 gives uniform continuity.
Not Lipschitz. Taking $y = 0$,

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \longrightarrow \infty$$

as $x \to 0^+$, so no bound M can hold for all $x \neq y$ in $[0, 1]$.

Problem 4.4.10. Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x), \qquad f(x)g(x), \qquad \frac{f(x)}{g(x)}, \qquad f(g(x))\,?$$

(Assume that the quotient and the composition are properly defined and thus at least continuous.)

Solution. Only the sum and the composition.
Sum: yes. Given $\epsilon > 0$, pick δ_1 for f and δ_2 for g, each answering $\epsilon/2$, and set $\delta = \min\{\delta_1, \delta_2\}$. For $x, y \in A$ with $|x - y| < \delta$,

$$\begin{aligned}|(f + g)(x) - (f + g)(y)| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

Composition: yes. Given $\epsilon > 0$, choose $\delta_1 > 0$ so that $|u - v| < \delta_1$ implies $|f(u) - f(v)| < \epsilon$, and then choose $\delta > 0$ so that $|x-y| < \delta$ implies $|g(x)-g(y)| < \delta_1$. Then $|x-y| < \delta$ gives $|f(g(x))-f(g(y))| < \epsilon$.
Product: no. Take $A = \mathbf{R}$ and $f(x) = g(x) = x$, which is uniformly continuous ($\delta = \epsilon$). The product is x^2, which fails to be uniformly continuous on $\mathbf{R}$: with $x_n = n + 1/n$ and $y_n = n$ we have $|x_n - y_n| \to 0$ while

$$|x_n^2 - y_n^2| = 2 + \tfrac{1}{n^2} \geq 2,$$

so Theorem 4.4.5 applies with $\epsilon_0 = 2$.
Quotient: no. Take $A = (0, 1)$, $f(x) = 1$ and $g(x) = x$; both are uniformly continuous on A and g never vanishes there. The quotient is $1/x$, and with $x_n = 1/n$, $y_n = 1/(n + 1)$ we get $|x_n - y_n| \to 0$ while

$$\left|\frac{1}{x_n} - \frac{1}{y_n}\right| = |n - (n + 1)| = 1,$$

so Theorem 4.4.5 applies with $\epsilon_0 = 1$.

4.6 Exercises 4.4.11–4.5.3

Problem 4.4.11. (Topological Characterization of Continuity). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Solution. Both directions are the ϵ–δ definition read through Theorem 4.3.2 (ii): continuity at c says every ϵ-neighborhood $V_\epsilon(g(c))$ pulls back to contain a δ-neighborhood $V_\delta(c)$.
($\Rightarrow$) Assume g is continuous and let O be open. Let $x \in g^{-1}(O)$, so $g(x) \in O$; since O is open there is $\epsilon > 0$ with $V_\epsilon(g(x)) \subseteq O$ (Definition 3.2.1). Continuity at x produces $\delta > 0$ with

$$|y - x| < \delta \implies |g(y) - g(x)| < \epsilon,$$

that is, $g(V_\delta(x)) \subseteq V_\epsilon(g(x)) \subseteq O$, and hence $V_\delta(x) \subseteq g^{-1}(O)$. So every point of $g^{-1}(O)$ is interior, and $g^{-1}(O)$ is open.
($\Leftarrow$) Assume preimages of open sets are open, and fix $c \in \mathbf{R}$ and $\epsilon > 0$. The set $O = V_\epsilon(g(c))$ is open, so $g^{-1}(O)$ is open; it contains c because $g(c) \in O$. Therefore there is $\delta > 0$ with $V_\delta(c) \subseteq g^{-1}(O)$, which says precisely

$$|x - c| < \delta \implies |g(x) - g(c)| < \epsilon.$$

Thus g is continuous at c, and c was arbitrary.

Problem 4.4.12. Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:
(a) $f^{-1}(B)$ is finite whenever B is finite.
(b) $f^{-1}(K)$ is compact whenever K is compact.
(c) $f^{-1}(A)$ is bounded whenever A is bounded.
(d) $f^{-1}(F)$ is closed whenever F is closed.

Solution. Only (d) is true; the constant function refutes (a), (b) and (c) simultaneously.
(a) False. Let $f(x) = 0$ for all x. Then $B = \{0\}$ is finite but $f^{-1}(B) = \mathbf{R}$.
(b) False. Same f: $K = \{0\}$ is compact (closed and bounded, Theorem 3.3.4) while $f^{-1}(K) = \mathbf{R}$ is unbounded, hence not compact.
(c) False. Same f and $A = \{0\}$: $f^{-1}(A) = \mathbf{R}$.
(d) True. Let F be closed. Then F^c is open (Theorem 3.2.13), so $f^{-1}(F^c)$ is open by Exercise 4.4.11. Because f is defined on all of $\mathbf{R}$, every x satisfies exactly one of $f(x) \in F$, $f(x) \in F^c$, so

$$f^{-1}(F^c) = \left(f^{-1}(F)\right)^c.$$

Thus $\left(f^{-1}(F)\right)^c$ is open, and $f^{-1}(F)$ is closed by Theorem 3.2.13 again.

Problem 4.4.13. (Continuous Extension Theorem).
(a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.
(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution. (a) Let $\epsilon > 0$ and take $\delta > 0$ from Definition 4.4.4. Since (x_n) is Cauchy there is N with $|x_n - x_m| < \delta$ for all $m, n \geq N$, and then

$$|f(x_n) - f(x_m)| < \epsilon \qquad \text{for all } m, n \geq N.$$

So $(f(x_n))$ is Cauchy (Definition 2.6.1).
(b) ($\Leftarrow$) If g extends to a continuous function on $[a, b]$, then that extension is continuous on a compact set (Theorem 3.3.4), hence uniformly continuous on $[a, b]$ by Theorem 4.4.7. A δ that answers ϵ for all pairs in $[a, b]$ answers it in particular for all pairs in (a, b), so the original g is uniformly continuous there.
($\Rightarrow$) Assume g is uniformly continuous on (a, b). Pick any sequence $(x_n) \subseteq (a, b)$ with $x_n \to a$. Convergent sequences are Cauchy (Theorem 2.6.2), so $(g(x_n))$ is Cauchy by part (a) and hence converges by the Cauchy Criterion (Theorem 2.6.4). Define

$$g(a) = \lim_{n\to\infty} g(x_n), \qquad g(b) = \lim_{n\to\infty} g(y_n),$$

the latter for any $(y_n) \subseteq (a, b)$ with $y_n \to b$.
To see the extension is continuous at a, let $\epsilon > 0$ and take $\delta > 0$ from uniform continuity answering $\epsilon/2$. Fix $x \in (a, b)$ with $|x - a| < \delta$. Since $x_n \to a$ and $g(x_n) \to g(a)$, choose n large enough that

$|x_n - a| < \delta$ and $|g(x_n) - g(a)| < \epsilon/2$. Then $|x - x_n| < \delta$ because both lie in $(a, a+\delta)$, so

$$
\begin{aligned}
|g(x) - g(a)| &\le |g(x) - g(x_n)| + |g(x_n) - g(a)| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Since $|g(a) - g(a)| = 0 < \epsilon$ as well, the same δ witnesses continuity of the extended g at a; the argument at b is identical. Continuity at the interior points is unaffected, so the extended g is continuous on $[a, b]$.

Problem 4.4.14. Construct an alternate proof of Theorem 4.4.7 using the open cover characterization of compactness from the Heine–Borel Theorem (Theorem 3.3.8 (iii)).

Solution. Fix $\epsilon > 0$; a finite subcover by *half*-neighborhoods produces the uniform δ. (Theorem 4.4.7: f continuous on a compact K is uniformly continuous on K.)
For each $x \in K$, continuity of f at x supplies $\delta(x) > 0$ with

$$
y \in K,\ |y - x| < \delta(x) \implies |f(y) - f(x)| < \tfrac{\epsilon}{2}.
$$

The half-radius neighborhoods

$$
\mathcal{O} = \left\{ V_{\delta(x)/2}(x) \ :\ x \in K \right\}
$$

form an open cover of K (each x lies in its own set). Since K is compact, Theorem 3.3.8 (iii) yields finitely many points $x_1, \ldots, x_n \in K$ with

$$
K \subseteq \bigcup_{i=1}^{n} V_{\delta(x_i)/2}(x_i),
$$

and we set

$$
\delta = \min\bigl\{\delta(x_1)/2, \ldots, \delta(x_n)/2\bigr\} > 0,
$$

positive precisely because the minimum is over a finite set.
Now let $y, z \in K$ satisfy $|y - z| < \delta$. Choose i with $y \in V_{\delta(x_i)/2}(x_i)$. Then $|y - x_i| < \delta(x_i)/2 < \delta(x_i)$, and

$$
|z - x_i| \le |z - y| + |y - x_i| < \delta + \tfrac{\delta(x_i)}{2} \le \delta(x_i).
$$

Both y and z therefore lie within $\delta(x_i)$ of x_i, so

$$
\begin{aligned}
|f(y) - f(z)| &\le |f(y) - f(x_i)| + |f(x_i) - f(z)| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Since δ was manufactured from ϵ alone, f is uniformly continuous on K by Definition 4.4.4.

Problem 4.5.1. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2 (Preservation of Connected Sets: if $f : G \to \mathbf{R}$ is continuous and $E \subseteq G$ is connected, then $f(E)$ is connected).

Solution. Apply Theorem 4.5.2 to $E = [a, b]$, connected by Theorem 3.4.7.
Let $f : [a, b] \to \mathbf{R}$ be continuous with $f(a) < L < f(b)$ (the case $f(a) > L > f(b)$ is the same with a and b exchanged). Whenever $x < w < y$ with $x, y \in [a, b]$ we have $w \in [a, b]$, so Theorem 3.4.7 makes $[a, b]$ connected, and Theorem 4.5.2 makes $f([a, b])$ connected. Since $f(a), f(b) \in f([a, b])$ and $f(a) < L < f(b)$, the converse half of Theorem 3.4.7 forces $L \in f([a, b])$: say $L = f(c)$ with $c \in [a, b]$. Finally $f(c) = L$ differs from both $f(a)$ and $f(b)$, so $c \in (a, b)$.

Problem 4.5.2. Provide an example of each of the following, or explain why the request is impossible.
(a) A continuous function defined on an open interval with range equal to a closed interval.
(b) A continuous function defined on a closed interval with range equal to an open interval.
(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from **R**.
(d) A continuous function defined on all of **R** with range equal to **Q**.

Solution. (a) Possible: $f(x) = \cos(4\pi x)$ on $(0,1)$. As x runs over $(0,1)$ the argument $4\pi x$ runs over $(0,4\pi)$, which contains π, 2π and 3π; hence $f(1/4) = -1$ and $f(1/2) = 1$ are attained, and the range is exactly $[-1,1]$.
(b) Impossible. A closed interval $[a,b]$ is compact, so Theorem 4.4.1 makes $f([a,b])$ compact, hence closed and bounded (Theorem 3.3.4). A nonempty open interval (c,d) is neither: if $d < \infty$ then d is a limit point of (c,d) outside it, and if $d = \infty$ the set is unbounded.
(c) Possible: $f(x) = \dfrac{1}{x(1-x)}$ on $(0,1)$, continuous there by Example 4.3.5 since the denominator never vanishes. From $x(1-x) \le 1/4$ we get $f \ge 4$, with $f(1/2) = 4$; and $f(x) \to \infty$ as $x \to 1^-$. Given $M \ge 4$, pick $x_0 \in (1/2, 1)$ with $f(x_0) \ge M$; Theorem 4.5.1 on $[1/2, x_0]$ attains M. So the range is exactly $[4,\infty)$.
(d) Impossible. **R** is connected, so by Theorem 4.5.2 the range $f(\mathbf{R})$ is a connected subset of **R**. But **Q** is disconnected: Example 3.4.5 (ii) writes

$$\begin{aligned} \mathbf{Q} &= A \cup B, \\ A &= \mathbf{Q} \cap (-\infty, \sqrt{2}), \\ B &= \mathbf{Q} \cap (\sqrt{2}, \infty) \end{aligned}$$

as a union of two nonempty separated sets. So $f(\mathbf{R}) = \mathbf{Q}$ is impossible.

Problem 4.5.3. A function f is *increasing* on A if $f(x) \le f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a,b]$.

Solution. Monotonicity turns a single value handed over by the intermediate value property into control on a whole one-sided neighborhood.
Fix $c \in [a,b]$ and $\varepsilon > 0$; we verify Theorem 4.3.2 (i) by building one-sided $\delta_\pm$ and taking $\delta = \min\{\delta_+, \delta_-\}$. To the right (take $\delta_+ = 1$ if $c = b$):
(i) $f(x) < f(c) + \varepsilon$ for every $x \in (c,b]$. Take $\delta_+ = 1$; monotonicity gives $f(c) \le f(x) < f(c) + \varepsilon$ there.
(ii) Some $y \in (c,b]$ has $f(y) \ge f(c) + \varepsilon$. Then $L = f(c) + \varepsilon/2$ lies strictly between $f(c)$ and $f(y)$, so Definition 4.5.3 applied to $c < y$ in $[a,b]$ supplies $d \in (c,y)$ with $f(d) = L$. Set $\delta_+ = d - c > 0$; for $c \le x < d$,

$$f(c) \le f(x) \le f(d) = f(c) + \tfrac{\varepsilon}{2} < f(c) + \varepsilon.$$

The left side mirrors this: if some $y \in [a,c)$ has $f(y) \le f(c) - \varepsilon$, Definition 4.5.3 on $[y,c]$ gives $d \in (y,c)$ with $f(d) = f(c) - \varepsilon/2$, and $\delta_- = c - d$ works because $d < x \le c$ forces $f(c) - \varepsilon < f(d) \le f(x) \le f(c)$; otherwise $\delta_- = 1$. So $x \in [a,b]$ with $|x - c| < \delta$ gives $|f(x) - f(c)| < \varepsilon$, for every $c \in [a,b]$.

4.7 Exercises 4.5.4–4.6.2

Problem 4.5.4. Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A : g(x) = g(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

Solution. One coincidence $g(x_1) = g(x_2)$ already breeds a continuum of them, so a nonempty F is uncountable. (The printed display writes f for the function the statement calls g; we use g throughout.)
Assume $F \neq \emptyset$: there are $x_1 < x_2$ in A with $g(x_1) = g(x_2)$, and $[x_1, x_2] \subseteq A$ because A is an interval.
(i) g constant on $[x_1, x_2]$. Every x there has a partner, so $[x_1, x_2] \subseteq F$, and a nondegenerate interval is uncountable (Theorem 1.6.1, transported by the bijection $t \mapsto x_1 + t(x_2 - x_1)$ of $(0, 1)$ onto (x_1, x_2)).
(ii) g not constant on $[x_1, x_2]$. Some point of $[x_1, x_2]$ carries a value other than $g(x_1)$, and it is neither endpoint (both have value $g(x_1)$), so there is $z \in (x_1, x_2)$ with $g(z) \neq g(x_1)$. Let J be the nondegenerate open interval with endpoints $g(x_1)$ and $g(z)$, and fix $L \in J$. Theorem 4.5.1 applied to the continuous g on $[x_1, z]$ gives

$$u(L) \in (x_1, z) \quad \text{with} \quad g(u(L)) = L;$$

since $g(x_2) = g(x_1)$, that same L lies strictly between $g(z)$ and $g(x_2)$, and Theorem 4.5.1 on $[z, x_2]$ gives

$$v(L) \in (z, x_2) \quad \text{with} \quad g(v(L)) = L.$$

Then $u(L) < z < v(L)$ with equal g-values, so $u(L) \in F$. The map $L \mapsto u(L)$ is injective, since $L = g(u(L))$ recovers L; hence $u(J) \subseteq F$ is in bijection with the uncountable interval J, and a countable F would make $u(J)$ countable by Theorem 1.5.7.

Problem 4.5.5. (a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously. (In the special case f continuous on $[a, b]$ with $f(a) < 0 < f(b)$, one sets $K = \{x \in [a, b] : f(x) \leq 0\}$ and $c = \sup K$, and must rule out $f(c) > 0$ and $f(c) < 0$.)
(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously. (With $L = 0$ and $f(a) < 0 < f(b)$, one puts $I_0 = [a, b]$, bisects at $z = (a + b)/2$, and keeps the half $I_1 = [a_1, b_1]$ on which f is negative at the left endpoint and nonnegative at the right; the construction is then repeated inductively.)

Solution. (a) $c = \sup K$ satisfies $f(c) = 0$; both alternatives are killed by the least-upper-bound property.
$K = \{x \in [a, b] : f(x) \leq 0\}$ contains a and is bounded above by b, so the Axiom of Completeness gives $c = \sup K \in [a, b]$.
(i) $f(c) > 0$. Continuity at c (Theorem 4.3.2 (i)) with $\varepsilon = f(c)$ gives $\delta > 0$ with $f(x) > 0$ for every $x \in (c - \delta, c + \delta) \cap [a, b]$. No such x lies in K, and no $x > c$ lies in K because c bounds K above; hence $c - \delta$ also bounds K above and $\sup K \leq c - \delta < c$, a contradiction.
(ii) $f(c) < 0$. Then $c < b$, since $f(b) > 0$. Continuity at c with $\varepsilon = -f(c) > 0$ gives $\delta > 0$ with $f(x) < 0$ for every $x \in (c - \delta, c + \delta) \cap [a, b]$. Any x_0 with $c < x_0 < \min\{c + \delta,\, b\}$ therefore lies in K and exceeds c, contradicting that c bounds K above.
Hence $f(c) = 0$, and $c \in (a, b)$ because $f(a) < 0 < f(b)$.
For the general statement of Theorem 4.5.1: if $f(a) < L < f(b)$, apply the above to $g = f - L$, which is continuous (Theorem 4.3.4) with $g(a) < 0 < g(b)$; if $f(a) > L > f(b)$, apply it to $g = L - f$.
(b) The nested intervals close down on the required point, and the Order Limit Theorem squeezes f there.
The construction produces $I_n = [a_n, b_n]$ with

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots, \qquad b_n - a_n = \frac{b - a}{2^n},$$

and, for every n,

$$f(a_n) < 0 \leq f(b_n).$$

By the Nested Interval Property (Theorem 1.4.1) there is a point $c \in \bigcap_{n=0}^{\infty} I_n$; in particular $c \in [a, b]$. Since $a_n, c \in I_n$,

$$|a_n - c| \le b_n - a_n = \frac{b-a}{2^n} \longrightarrow 0,$$

so $(a_n) \to c$, and identically $(b_n) \to c$. Continuity of f at c and the sequential characterization Theorem 4.3.2 (iii) give

$$f(a_n) \to f(c) \qquad \text{and} \qquad f(b_n) \to f(c).$$

Applying the Order Limit Theorem (Theorem 2.3.4) to $f(a_n) < 0$ and to $f(b_n) \ge 0$ yields

$$f(c) \le 0 \qquad \text{and} \qquad f(c) \ge 0,$$

so $f(c) = 0$. As in (a), $c \ne a$ and $c \ne b$, so $c \in (a, b)$, and the general case follows by replacing f with $f - L$ or $L - f$.

Problem 4.5.6. Let $f : [0, 1] \to \mathbf{R}$ be continuous with $f(0) = f(1)$.
(a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.
(b) Show that for each $n \in \mathbf{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.
(c) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

Solution. (a) Apply the Intermediate Value Theorem to $g(x) = f(x + 1/2) - f(x)$ on $[0, 1/2]$. g is continuous there by Theorem 4.3.4 (and Theorem 4.3.9 for the composition with $x \mapsto x + 1/2$), and

$$\begin{aligned} g\left(\tfrac{1}{2}\right) &= f(1) - f\left(\tfrac{1}{2}\right) \\ &= f(0) - f\left(\tfrac{1}{2}\right) = -g(0). \end{aligned}$$

If $g(0) = 0$, take $x = 0$, $y = 1/2$. Otherwise $g(0)$ and $g(1/2)$ are nonzero of opposite sign, so 0 lies strictly between them and Theorem 4.5.1 gives $c \in (0, 1/2)$ with $g(c) = 0$; take $x = c$, $y = c + 1/2$.
(b) Same device with step $1/n$, plus a telescoping sum.
Let $g(x) = f(x + 1/n) - f(x)$, continuous on $[0, 1 - 1/n]$. Then

$$\begin{aligned} \sum_{k=0}^{n-1} g\left(\frac{k}{n}\right) &= \sum_{k=0}^{n-1} \left[f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right)\right] \\ &= f(1) - f(0) = 0. \end{aligned}$$

If some term vanishes, say $g(k/n) = 0$, take $x_n = k/n$ and $y_n = (k + 1)/n$. Otherwise the n nonzero terms sum to zero, so they cannot all have the same sign: there are indices k, l with

$$g\left(\frac{k}{n}\right) > 0 > g\left(\frac{l}{n}\right).$$

Both k/n and l/n lie in $[0, 1 - 1/n]$, so applying Theorem 4.5.1 to g on the closed interval with these two endpoints produces c strictly between them with $g(c) = 0$. Take $x_n = c$, $y_n = c + 1/n$; then $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.
(c) Take

$$f(x) = \sin^2\left(\frac{5\pi x}{2}\right) - x.$$

This is continuous on $[0, 1]$, and $f(0) = 0$, $f(1) = \sin^2(5\pi/2) - 1 = 1 - 1 = 0$, so $f(0) = f(1)$. For any x with $x, x + 2/5 \in [0, 1]$,

$$\begin{aligned} f\left(x + \tfrac{2}{5}\right) - f(x) &= \sin^2\left(\tfrac{5\pi x}{2} + \pi\right) - \sin^2\left(\tfrac{5\pi x}{2}\right) - \tfrac{2}{5} \\ &= -\frac{2}{5}, \end{aligned}$$

since $\sin(\theta + \pi) = -\sin\theta$ makes $\sin^2$ periodic with period π. Hence $f(x) \ne f(y)$ whenever $|x - y| = 2/5$ with $x, y \in [0, 1]$.

Problem 4.5.7. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Solution. Apply the Intermediate Value Theorem to $g(x) = f(x) - x$.
g is continuous on $[0, 1]$ by Theorem 4.3.4, and because f takes values in $[0, 1]$,

$$g(0) = f(0) - 0 \geq 0, \qquad g(1) = f(1) - 1 \leq 0.$$

(i) If $g(0) = 0$ then $x = 0$ is a fixed point; if $g(1) = 0$ then $x = 1$ is.
(ii) Otherwise $g(1) < 0 < g(0)$, so 0 lies strictly between $g(0)$ and $g(1)$ and Theorem 4.5.1 supplies $c \in (0, 1)$ with $g(c) = 0$, i.e. $f(c) = c$.

Problem 4.5.8. (Inverse functions). If a function $f : A \to \mathbf{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.
Show that if f is continuous on an interval $[a, b]$ and one-to-one, then f^{-1} is also continuous.

Solution. Sequentially: if $y_n \to y$, every subsequential limit of $\big(f^{-1}(y_n)\big)$ equals $f^{-1}(y)$, because f is continuous and one-to-one.
Write $g := f^{-1}$. The Extreme Value Theorem (Theorem 4.4.2, applicable since $[a, b]$ is compact and f is continuous on it) gives a minimum c and a maximum d of f, attained at some $p, q \in [a, b]$; the Intermediate Value Theorem (Theorem 4.5.1) applied to f on the interval with endpoints p and q attains every value in between. Hence

$$g : [c, d] \longrightarrow [a, b], \qquad f([a, b]) = [c, d].$$

Fix $y \in [c, d]$ and $(y_n) \subseteq [c, d]$ with $y_n \to y$, and put

$$x_n = g(y_n) \in [a, b], \qquad x = g(y) \in [a, b],$$

so $f(x_n) = y_n$ and $f(x) = y$. If $x_n \not\to x$, there are $\epsilon_0 > 0$ and a subsequence (x_{n_k}) with

$$|x_{n_k} - x| \geq \epsilon_0 \quad \text{for all } k.$$

This subsequence is bounded (it lies in $[a, b]$), so Bolzano-Weierstrass (Theorem 2.5.5) supplies $x_{n_{k_l}} \to x'$, with $x' \in [a, b]$ because $[a, b]$ is closed; the displayed inequality passes to the limit, giving $|x' - x| \geq \epsilon_0$ and hence $x' \neq x$. But f is continuous at x', so by Theorem 4.3.2 (iii), together with Theorem 2.5.2 for the subsequence of (y_n),

$$\begin{aligned} f(x') &= \lim_{l \to \infty} f\big(x_{n_{k_l}}\big) \\ &= \lim_{l \to \infty} y_{n_{k_l}} = y = f(x), \end{aligned}$$

contradicting injectivity. Therefore $x_n \to x$, i.e. $g(y_n) \to g(y)$ for every sequence $y_n \to y$ in $[c, d]$, and Theorem 4.3.2 (iii), now applied to g, makes $g = f^{-1}$ continuous at every $y \in [c, d]$.
Method (2): f is strictly monotone. Say $f(a) < f(b)$. Then $f(a) < f(t) < f(b)$ for every $t \in (a, b)$: if $f(t) > f(b)$, Theorem 4.5.1 on $[a, t]$ returns $x_1 \in (a, t)$ with $f(x_1) = f(b)$, and if $f(t) < f(a)$ it returns $x_2 \in (t, b)$ with $f(x_2) = f(a)$; either one contradicts injectivity. Since this conclusion needs only $f(a) < f(b)$, it may be re-applied on $[t, b]$, where $f(t) < f(b)$, giving $f(t) < f(s)$ for $t < s < b$; so f is strictly increasing on $[a, b]$. (If $f(a) > f(b)$, run this on $-f$, so f is strictly decreasing.)
Take f increasing. Then g is increasing on $[c, d]$ with $g([c, d]) = [a, b]$, so g has the intermediate value property of Definition 4.5.3: for $y_1 < y_2$ in $[c, d]$ and L strictly between $g(y_1)$ and $g(y_2)$, the point $y_0 = f(L)$ lies in (y_1, y_2) and $g(y_0) = L$. Exercise 4.5.3 (increasing plus the intermediate value property gives continuity) finishes. If f decreases, $-f$ increases with inverse $y \mapsto g(-y)$, so g is continuous by Theorem 4.3.9.

Problem 4.6.1. Given a function $f : \mathbf{R} \to \mathbf{R}$, define $D_f \subseteq \mathbf{R}$ to be the set of points where f fails to be continuous. In Section 4.1 we saw that Dirichlet's function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has $D_g = \mathbf{R}$; the modification

$$h(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has $D_h = \mathbf{R} \setminus \{0\}$, zero being the only point of continuity; and Thomae's function $t(x)$ has $D_t = \mathbf{Q}$. Using modifications of these functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that
(a) $D_f = \mathbf{Z}^c$.
(b) $D_f = \{x : 0 < x \leq 1\}$.

Solution. The template is $f = \varphi \cdot \mathbf{1_Q}$ for a continuous φ: such an f is continuous exactly where φ vanishes. If $\varphi(c) = 0$ then $|f(x) - f(c)| = |f(x)| \leq |\varphi(x)| \to 0$; if $\varphi(c) \neq 0$, rationals $x_n \to c$ give $f(x_n) = \varphi(x_n) \to \varphi(c)$ while irrationals $y_n \to c$ give $f(y_n) \to 0$, so one of these two sequences violates Corollary 4.3.3. Hence

$$D_{\varphi \cdot \mathbf{1_Q}} = \{x \in \mathbf{R} : \varphi(x) \neq 0\}.$$

(a) Take

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Since $\sin(\pi x)$ is continuous and vanishes precisely on $\mathbf{Z}$, the displayed identity gives $D_f = \mathbf{Z}^c$.
(b) The zero set of a continuous φ is closed, so no such template yields the half-open $(0, 1]$; break the template at the right endpoint instead. Take

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \cap (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

For $c < 0$ or $c > 1$ the function vanishes on a whole neighborhood of c, and at $c = 0$ we have $|f(x) - f(0)| \leq |x|$, so $\delta = \epsilon$ serves in Theorem 4.3.2 (i). For $0 < c < 1$, rationals $x_n \to c$ lying in $(0, 1)$ give $f(x_n) \to c \neq 0$ while irrationals give $f \to 0$; and at $c = 1$, irrationals $y_n \to 1$ give $f(y_n) = 0 \neq 1 = f(1)$. Corollary 4.3.3 applies at each such c, so

$$D_f = \{x : 0 < x \leq 1\}.$$

Problem 4.6.2. Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Solution. $D_f = A$.
$A \subseteq D_f$: fix a_n. Since A is countable it contains no interval, so A^c is dense and we may pick $x_k \in A^c$ with $x_k \to a_n$. Then $f(x_k) = 0$ for every k, while $f(a_n) = 1/n \neq 0$, so f is discontinuous at a_n by Corollary 4.3.3.
$A^c \subseteq D_f^c$: fix $c \notin A$, so $f(c) = 0$, and let $\epsilon > 0$. Choose $N \in \mathbf{N}$ with $1/N < \epsilon$ (Archimedean Property, Theorem 1.4.2) and set

$$\delta = \min\{|c - a_1|, |c - a_2|, \ldots, |c - a_N|\} > 0,$$

positive because it is a minimum of finitely many nonzero numbers ($c \neq a_n$ for all n). If $|x - c| < \delta$ then $x \notin \{a_1, \ldots, a_N\}$, so either $x \notin A$ and $f(x) = 0$, or $x = a_m$ with $m > N$ and $f(x) = 1/m < 1/N < \epsilon$. In both cases

$$|f(x) - f(c)| = |f(x)| < \epsilon,$$

so f is continuous at c by Theorem 4.3.2 (i).

4.8 Exercises 4.6.3–4.6.9

Problem 4.6.3. Definition 4.6.2 states that, given a limit point c of a set A and a function $f : A \to \mathbf{R}$, we write

$$\lim_{x\to c^+} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$; equivalently, in terms of sequences, $\lim_{x\to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n > c$ and $\lim(x_n) = c$.
State a similar definition for the left-hand limit

$$\lim_{x\to c^-} f(x) = L.$$

Solution. Given a limit point c of A and $f : A \to \mathbf{R}$, write $\lim_{x\to c^-} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < c - x < \delta$$

(with $x \in A$, as always): the neighborhood $(c, c + \delta)$ of Definition 4.6.2 is replaced by $(c - \delta, c)$. Equivalently, in terms of sequences, $\lim_{x\to c^-} f(x) = L$ if $\lim f(x_n) = L$ for all sequences $(x_n) \subseteq A$ satisfying $x_n < c$ and $\lim(x_n) = c$.

Problem 4.6.4. Supply a proof for the following proposition.
Theorem 4.6.3. Given $f : A \to \mathbf{R}$ and a limit point c of A, $\lim_{x\to c} f(x) = L$ if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Solution. Both directions are the observation that the deleted neighborhood $\{x : 0 < |x - c| < \delta\}$ is the disjoint union of $(c - \delta, c)$ and $(c, c + \delta)$.
($\Rightarrow$) Let $\epsilon > 0$ and take $\delta > 0$ from Definition 4.2.1 so that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. Since

$$0 < x - c < \delta \;\Rightarrow\; 0 < |x - c| < \delta, \qquad 0 < c - x < \delta \;\Rightarrow\; 0 < |x - c| < \delta,$$

the same δ witnesses both one-sided limits.
($\Leftarrow$) Let $\epsilon > 0$. Pick $\delta_1 > 0$ with $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta_1$, and $\delta_2 > 0$ with $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\} > 0$. If $0 < |x - c| < \delta$ then $x \neq c$, so exactly one of

$$0 < x - c < \delta \le \delta_1 \qquad \text{or} \qquad 0 < c - x < \delta \le \delta_2$$

holds, and in either case $|f(x) - L| < \epsilon$. Hence $\lim_{x\to c} f(x) = L$.
(If c fails to be a limit point of A from one side, the corresponding one-sided condition holds vacuously for every L, and the argument is unaffected.)

Problem 4.6.5. Discontinuities are divided into three categories:
(i) If $\lim_{x\to c} f(x)$ exists but has a value different from $f(c)$, the discontinuity at c is called removable.
(ii) If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then f has a jump discontinuity at c.
(iii) If $\lim_{x\to c} f(x)$ does not exist for some other reason, then the discontinuity at c is called an essential discontinuity.
Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution. At every c a monotone f has both one-sided limits, and they straddle $f(c)$; this rules out (i) and (iii) at once. Assume $f : \mathbf{R} \to \mathbf{R}$ is increasing (Definition 4.6.1); the decreasing case follows by applying the result to $-f$, which is increasing and has the same discontinuities.

Fix $c \in \mathbf{R}$ and set

$$L^- = \sup\{f(x) : x < c\}, \qquad L^+ = \inf\{f(x) : x > c\}.$$

Both sets are nonempty, and $f(c)$ is an upper bound for the first and a lower bound for the second, so L^- and L^+ exist in $\mathbf{R}$ by the Axiom of Completeness and its greatest-lower-bound form (Exercise 1.3.3), and

$$L^- \leq f(c) \leq L^+.$$

Moreover $\lim_{x\to c^-} f(x) = L^-$: given $\epsilon > 0$, Lemma 1.3.8 supplies $x_0 < c$ with $f(x_0) > L^- - \epsilon$, and then $\delta = c - x_0 > 0$ works, since $0 < c - x < \delta$ forces $x_0 < x < c$ and monotonicity gives

$$L^- - \epsilon < f(x_0) \leq f(x) \leq L^-.$$

The symmetric argument (Lemma 1.3.8 for infima, Exercise 1.3.1 (b)) gives $\lim_{x\to c^+} f(x) = L^+$.
Both one-sided limits therefore exist, so category (iii) is impossible. If in addition $L^- = L^+$, then Theorem 4.6.3 says $\lim_{x\to c} f(x)$ exists and equals that common value, which by the straddling inequality is forced to be $f(c)$:

$$L^- \leq f(c) \leq L^+ = L^- \;\Rightarrow\; \lim_{x\to c} f(x) = f(c),$$

so f is continuous at c and category (i) is impossible as well. Hence every discontinuity of f has $L^- \neq L^+$, i.e. is a jump discontinuity.

Problem 4.6.6. Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Solution. Assign to each jump the rational number sitting inside the gap it opens. Take $f : \mathbf{R} \to \mathbf{R}$ increasing (replace f by $-f$ if decreasing), and for each c let

$$L^-(c) = \sup_{x<c} f(x), \qquad L^+(c) = \inf_{x>c} f(x),$$

the one-sided limits produced in Exercise 4.6.5. By that exercise $D_f = \{c : L^-(c) < L^+(c)\}$, so for $c \in D_f$ the open interval $J_c = (L^-(c), L^+(c))$ is nonempty, and by the Density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) we may choose $q(c) \in J_c \cap \mathbf{Q}$.
The map $q : D_f \to \mathbf{Q}$ is injective, because the intervals J_c are pairwise disjoint. Indeed, let $c_1 < c_2$ lie in D_f and pick any x with $c_1 < x < c_2$. Then x belongs to $\{t : t > c_1\}$ and to $\{t : t < c_2\}$, so

$$L^+(c_1) \leq f(x) \leq L^-(c_2),$$

and therefore every point of J_{c_1} is $< L^+(c_1) \leq L^-(c_2) <$ every point of J_{c_2}. Hence $q(c_1) \neq q(c_2)$.
Being injective, q is a bijection from D_f onto its range $q(D_f) \subseteq \mathbf{Q}$, which is the required bijection with a subset of $\mathbf{Q}$. Since $\mathbf{Q}$ is countable (Theorem 1.5.6 (i)), Theorem 1.5.7 makes $q(D_f)$ either finite or countable, and a set in bijective correspondence with a finite or countable set is finite or countable. Thus D_f is finite or countable, never uncountable.

Problem 4.6.7. Definition 4.6.4. A set that can be written as the countable union of closed sets is in the class F_σ.
In Section 4.1 we constructed functions where the set of discontinuity was $\mathbf{R}$ (Dirichlet's function), $\mathbf{R} \setminus \{0\}$ (modified Dirichlet function), and $\mathbf{Q}$ (Thomae's function).
(a) Show that in each of the above cases we get an F_σ set as the set where the function is discontinuous.
(b) Show that the two sets of discontinuity in Exercise 4.6.1 are F_σ sets.

Solution. (a) Exhibit the unions.

$$\mathbf{R} = \bigcup_{n=1}^{\infty} [-n, n],$$
$$\mathbf{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \Big([-n, -\tfrac{1}{n}] \cup [\tfrac{1}{n}, n]\Big),$$
$$\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}.$$

Closed intervals are closed (Example 3.2.9 (ii)), a finite union of closed sets is closed (Theorem 3.2.14 (i)), and singletons are closed (a one-point set has no limit points, so Definition 3.2.7 holds vacuously); the third union is indexed by the countable set $\mathbf{Q}$ (Theorem 1.5.6 (i)). For the middle line, $x \neq 0$ gives $1/|x| < n$ and $|x| < n$ for some n by the Archimedean Property (Theorem 1.4.2), so x lies in the nth set; conversely no set in the union contains 0.
(b) Every open interval is already a countable union of closed intervals:

$$(a, b) = \bigcup_{n=N}^{\infty} \Big[a + \tfrac{1}{n},\ b - \tfrac{1}{n}\Big], \qquad 2/N < b - a,$$

such an N existing by Theorem 1.4.2. Each $x \in (a, b)$ satisfies $1/n < \min\{x - a, b - x\}$ for all large n, so x lies in the union; the reverse inclusion is clear. Hence for part (a) of Exercise 4.6.1,

$$\mathbf{Z}^c = \bigcup_{k \in \mathbf{Z}} (k, k + 1)$$

is a countable union of countable unions of closed sets, which is again a countable union of closed sets (Theorem 1.5.8 (ii)), so $\mathbf{Z}^c \in F_\sigma$. For part (b),

$$\{x : 0 < x \leq 1\} = \{1\} \cup \bigcup_{n=3}^{\infty} \Big[\tfrac{1}{n},\ 1 - \tfrac{1}{n}\Big]$$

is a countable union of closed sets outright, hence in F_σ.

Problem 4.6.8. Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.
(Definition 4.6.5: for f defined on $\mathbf{R}$, f is α-continuous at x if there is a $\delta > 0$ with $|f(y) - f(z)| < \alpha$ for all $y, z \in (x - \delta, x + \delta)$; and $D_f^\alpha = \{x \in \mathbf{R} : f \text{ is not } \alpha\text{-continuous at } x\}$.)

Solution. The complement is open, because a δ that witnesses α-continuity at x witnesses it at every point of $(x - \delta, x + \delta)$.
Let $x \notin D_f^\alpha$ and pick $\delta > 0$ as in Definition 4.6.5, so that

$$|f(y) - f(z)| < \alpha \quad \text{for all } y, z \in (x - \delta, x + \delta).$$

Given $w \in (x - \delta, x + \delta)$, set $\delta_w = \delta - |w - x| > 0$. If $|y - w| < \delta_w$ then

$$|y - x| \leq |y - w| + |w - x| < \delta_w + |w - x| = \delta,$$

so $(w - \delta_w, w + \delta_w) \subseteq (x - \delta, x + \delta)$ and the displayed estimate applies to every pair $y, z \in (w - \delta_w, w + \delta_w)$. Hence f is α-continuous at w, i.e. $w \notin D_f^\alpha$.
Thus $(D_f^\alpha)^c$ contains a neighborhood of each of its points and so is open, and D_f^α is closed by Theorem 3.2.13.

Problem 4.6.9. If $\alpha < \alpha'$, show that $D_f^{\alpha'} \subseteq D_f^{\alpha}$.

Solution. A δ that keeps the variation below α keeps it below the larger α', so α-continuity implies α'-continuity; take contrapositives.
In detail, suppose $x \notin D_f^{\alpha}$, and let $\delta > 0$ be as in Definition 4.6.5 for α. For all $y, z \in (x - \delta, x + \delta)$,

$$|f(y) - f(z)| < \alpha < \alpha',$$

so the same δ shows f is α'-continuous at x, i.e. $x \notin D_f^{\alpha'}$. Therefore $x \in D_f^{\alpha'}$ forces $x \in D_f^{\alpha}$, which is the asserted inclusion.

4.9 Exercises 4.6.10–4.6.11

Problem 4.6.10. Let $\alpha > 0$ be given. Show that if f is continuous at x, then it is α-continuous at x as well. Explain how it follows that $D_f^{\alpha} \subseteq D_f$.

Solution. Run the definition of continuity with $\varepsilon = \alpha/2$ and use the triangle inequality through $f(x)$.
Since f is continuous at x (Definition 4.3.1) and $\alpha/2 > 0$, there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \alpha/2$. Then for any $y, z \in (x - \delta, x + \delta)$,

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \\ &< \tfrac{\alpha}{2} + \tfrac{\alpha}{2} = \alpha. \end{aligned}$$

So this δ meets the requirement of Definition 4.6.5 and f is α-continuous at x.
For the inclusion, take contrapositives of what was just proved: if $x \notin D_f$, i.e. f is continuous at x, then f is α-continuous at x, i.e. $x \notin D_f^{\alpha}$. Hence $D_f^{\alpha} \subseteq D_f$.

Problem 4.6.11. Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n},$$

where $\alpha_n = 1/n$.

Solution. Take $\alpha = \varepsilon_0$, the ε for which continuity fails at x.
Failure of continuity at x (the negation of Definition 4.3.1) supplies an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is a point y_δ with

$$|y_\delta - x| < \delta \quad \text{and} \quad |f(y_\delta) - f(x)| \geq \varepsilon_0.$$

Put $\alpha = \varepsilon_0$. Given any $\delta > 0$, the two points y_δ, x both lie in $(x - \delta, x + \delta)$ yet $|f(y_\delta) - f(x)| \geq \alpha$, so no δ can satisfy Definition 4.6.5. Thus f is not α-continuous at x, i.e. $x \in D_f^{\varepsilon_0}$.
Now the two inclusions. If $x \in D_f$, produce $\varepsilon_0 > 0$ as above and use the Archimedean Property (Theorem 1.4.2) to choose $n \in \mathbf{N}$ with $\alpha_n = 1/n < \varepsilon_0$. Exercise 4.6.9, applied with $\alpha = 1/n < \alpha' = \varepsilon_0$, gives

$$x \in D_f^{\varepsilon_0} \subseteq D_f^{1/n} = D_f^{\alpha_n},$$

so $D_f \subseteq \bigcup_{n=1}^{\infty} D_f^{\alpha_n}$. Conversely $D_f^{\alpha_n} \subseteq D_f$ for every n by Exercise 4.6.10, so the union is contained in D_f. Hence

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

5 The Derivative

Contents

5.1 Exercises 5.2.1–5.2.7

Problem 5.2.1. Supply proofs for parts (i) and (ii) of Theorem 5.2.4.
(Theorem 5.2.4, Algebraic Differentiability Theorem: let f and g be functions defined on an interval A, both differentiable at $c \in A$. Then (i) $(f+g)'(c) = f'(c) + g'(c)$, and (ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbf{R}$.)

Solution. Both are the Algebraic Limit Theorem for Functional Limits (Corollary 4.2.4) applied to difference quotients, whose limits exist by hypothesis.
(i) For $x \in A$ with $x \neq c$,

$$\frac{(f+g)(x) - (f+g)(c)}{x-c} = \frac{f(x) - f(c)}{x-c} + \frac{g(x) - g(c)}{x-c}.$$

Each summand on the right tends to $f'(c)$, respectively $g'(c)$, as $x \to c$ (Definition 5.2.1), so both limits exist and Corollary 4.2.4 (ii) gives

$$(f+g)'(c) = \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} = f'(c) + g'(c).$$

(ii) For $x \neq c$,

$$\frac{(kf)(x) - (kf)(c)}{x-c} = k \cdot \frac{f(x) - f(c)}{x-c},$$

and Corollary 4.2.4 (i) lets the constant k pass through the limit, so the limit exists and

$$(kf)'(c) = k \lim_{x \to c} \frac{f(x) - f(c)}{x-c} = kf'(c).$$

Problem 5.2.2. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbf{R}$.
(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.
(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.
(c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.
(d) A function f differentiable at zero but not differentiable at any other point.

Solution. (c) is the impossible one.
(a) Possible: $f(x) = g(x) = |x|$. Neither is differentiable at zero (Example 5.2.2 (ii)), while $(fg)(x) = x^2$ has $(fg)'(0) = 0$ by Example 5.2.2 (i).
(b) Possible: $f(x) = |x|$ and $g(x) = 0$. Then g is differentiable at zero and fg is identically zero, hence differentiable at zero.
(c) Impossible. If $f+g$ and g are both differentiable at zero, then so is

$$f = (f+g) + (-1)g,$$

by Theorem 5.2.4 (i) and (ii), contradicting the requirement that f not be differentiable at zero.
(d) Possible: take

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

At zero, $|f(x) - f(0)|/|x-0| = |f(x)|/|x| \leq |x|$ for all $x \neq 0$, so the difference quotient is squeezed to 0 and $f'(0) = 0$. At any $c \neq 0$, f fails to be continuous: rationals $x_n \to c$ give $f(x_n) = x_n^2 \to c^2$ and irrationals $y_n \to c$ give $f(y_n) \to 0$, and since $c^2 \neq 0$ at least one of these limits differs from $f(c)$, so Corollary 4.3.3 applies. By Theorem 5.2.3, f is therefore not differentiable at any $c \neq 0$.

Problem 5.2.3. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.
(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.
(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).
(Theorem 5.2.4 (iv): if f, g are differentiable at c and $g(c) \neq 0$, then $(f/g)'(c) = \big(g(c)f'(c) - f(c)g'(c)\big)/[g(c)]^2$.)

Solution. (a) $h'(c) = -1/c^2$ on any interval A with $0 \notin A$. Indeed, for $x, c \in A$ with $x \neq c$,

$$\frac{h(x) - h(c)}{x - c} = \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \frac{c - x}{xc\,(x - c)} = \frac{-1}{xc},$$

and $\lim_{x \to c}(-1/(xc)) = -1/c^2$ by Corollary 4.2.4, since $c \neq 0$.
(b) Since g is differentiable at c it is continuous there (Theorem 5.2.3), so with $\varepsilon = |g(c)|/2$ there is $\delta > 0$ with $g(x) \in B := V_{|g(c)|/2}(g(c))$ for every x in the interval $A' := V_\delta(c) \cap A$. Here B is an interval each of whose points has modulus exceeding $|g(c)|/2$, so $0 \notin B$ and $h(x) = 1/x$ is defined on B; restricting to A' changes no derivative at c. The Chain Rule (Theorem 5.2.5) applied to $h \circ g$ with part (a) gives

$$\left(\frac{1}{g}\right)'(c) = h'(g(c))\, g'(c) = \frac{-g'(c)}{[g(c)]^2}.$$

Now $f/g = f \cdot (1/g)$, so the Product Rule, Theorem 5.2.4 (iii), yields

$$\begin{aligned}\left(\frac{f}{g}\right)'(c) &= f'(c)\,\frac{1}{g(c)} + f(c)\,\frac{-g'(c)}{[g(c)]^2} \\ &= \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.\end{aligned}$$

(c) Keep the interval A' from (b), on which g never vanishes. For $x \in A'$, $x \neq c$, add and subtract $f(c)g(c)$ in the numerator:

$$\begin{aligned}\frac{(f/g)(x) - (f/g)(c)}{x - c} &= \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)\,(x - c)} \\ &= \frac{g(c)\big(f(x) - f(c)\big) - f(c)\big(g(x) - g(c)\big)}{g(x)g(c)\,(x - c)} \\ &= \frac{1}{g(x)g(c)}\left[g(c)\frac{f(x) - f(c)}{x - c} - f(c)\frac{g(x) - g(c)}{x - c}\right].\end{aligned}$$

As $x \to c$ the two difference quotients tend to $f'(c)$ and $g'(c)$, and $g(x) \to g(c) \neq 0$ by Theorem 5.2.3, so Corollary 4.2.4 (its quotient part (iv) is legitimate because the limit of $g(x)g(c)$ is $[g(c)]^2 \neq 0$) gives

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

Problem 5.2.4. Follow these steps to provide a slightly modified proof of the Chain Rule.
(a) Show that a function $h : A \to \mathbf{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbf{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x - a) \qquad \text{for all } x \in A.$$

(b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5 (the Chain Rule:

if $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ satisfy $f(A) \subseteq B$, f is differentiable at $c \in A$, and g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c))$.

Solution. (a) The witness is the difference quotient itself, filled in at a by the derivative:

$$l(x) = \begin{cases} \dfrac{h(x) - h(a)}{x - a} & \text{if } x \neq a, \\ h'(a) & \text{if } x = a. \end{cases}$$

($\Rightarrow$) Assume $h'(a)$ exists and define l as above. The identity $h(x) - h(a) = l(x)(x - a)$ holds for $x \neq a$ by construction and at $x = a$ because both sides are 0. Continuity at a is exactly Definition 5.2.1: $\lim_{x\to a} l(x) = h'(a) = l(a)$.
($\Leftarrow$) Assume such an l exists. Dividing the identity by $x - a$ gives

$$\frac{h(x) - h(a)}{x - a} = l(x) \quad \text{for all } x \in A,\ x \neq a,$$

and continuity of l at a forces $\lim_{x\to a} l(x) = l(a)$. Hence the limit defining $h'(a)$ exists and equals $l(a)$.
(b) Apply ($\Rightarrow$) twice. Since f is differentiable at c, there is $l : A \to \mathbf{R}$, continuous at c with $l(c) = f'(c)$, such that

$$f(x) - f(c) = l(x)(x - c) \qquad \text{for all } x \in A.$$

Since g is differentiable at $f(c)$, there is $m : B \to \mathbf{R}$, continuous at $f(c)$ with $m(f(c)) = g'(f(c))$, such that

$$g(y) - g(f(c)) = m(y)\big(y - f(c)\big) \qquad \text{for all } y \in B.$$

Because $f(A) \subseteq B$, we may substitute $y = f(x)$ for any $x \in A$ and then use the first identity:

$$\begin{aligned} (g \circ f)(x) - (g \circ f)(c) &= m(f(x))\big(f(x) - f(c)\big) \\ &= \underbrace{m(f(x))\, l(x)}_{=:\, L(x)} (x - c), \end{aligned}$$

valid for all $x \in A$. It remains to see that L is continuous at c. The function f is continuous at c (Theorem 5.2.3) and m is continuous at $f(c)$, so $m \circ f$ is continuous at c by Theorem 4.3.9; l is continuous at c; and products of functions continuous at a point are continuous there (Theorem 4.3.4). By the ($\Leftarrow$) direction of part (a), $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = L(c) = m(f(c))\, l(c) = g'(f(c))\, f'(c).$$

Problem 5.2.5. Let

$$f_a(x) = \begin{cases} x^a & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(a) For which values of a is f continuous at zero?
(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
(c) For which values of a is f twice-differentiable?

Solution. (a) $a > 0$. The left-hand limit is 0 for every a, so continuity at zero is the statement $\lim_{x\to 0^+} x^a = 0$. For $a > 0$ this holds, since given $\varepsilon > 0$ the choice $\delta = \varepsilon^{1/a}$ gives $0 < x < \delta \Rightarrow x^a < \varepsilon$. For $a = 0$ the right-hand limit is $1 \neq 0$, and for $a < 0$ we have $x^a \to +\infty$; in both cases f_a is discontinuous at zero.
(b) $a > 1$, and then f_a' is continuous on all of $\mathbf{R}$. The difference quotient at zero is

$$\frac{f_a(x) - f_a(0)}{x - 0} = \begin{cases} x^{a-1} & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

whose left-hand limit is 0. So the limit exists exactly when $\lim_{x\to 0^+} x^{a-1} = 0$, which by the computation in (a) happens exactly when $a-1>0$; then $f_a'(0)=0$. (For $a=1$ the right-hand limit is $1\neq 0$, and for $a<1$ it is $+\infty$.) For such a,

$$f_a'(x) = \begin{cases} ax^{a-1} & \text{if } x>0,\\ 0 & \text{if } x\le 0,\end{cases}$$

that is, $f_a' = a\,f_{a-1}$. Since $a-1>0$, part (a) says f_{a-1} is continuous at zero, and away from zero f_a' is continuous by the usual rules. So yes: whenever f_a is differentiable at zero, f_a' is continuous.
(c) $a>2$. By (b), $f_a' = af_{a-1}$, and this is differentiable at zero precisely when f_{a-1} is, i.e. precisely when $a-1>1$. (At $a=2$, $f_2'(x)=2x$ for $x>0$ and 0 for $x\le 0$, whose difference quotient at zero has right-hand limit 2 and left-hand limit 0.) Away from zero $f_a''(x) = a(a-1)x^{a-2}$ for $x>0$ and 0 for $x<0$ always exists, so f_a is twice-differentiable on $\mathbf{R}$ exactly for $a>2$.

Problem 5.2.6. Let g be defined on an interval A, and let $c\in A$.
(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}.$$

(b) Assume A is open. If g is differentiable at $c\in A$, show

$$g'(c) = \lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h}.$$

Solution. (a) The substitution $h=x-c$ carries one ε-δ statement to the other verbatim, so the two limits exist together and agree. Write $F(x) = \big(g(x)-g(c)\big)/(x-c)$ for $x\in A$, $x\neq c$, and $G(h)=F(c+h)$, defined for exactly those $h\neq 0$ with $c+h\in A$. Fix $L\in\mathbf{R}$. By Definition 4.2.1, $\lim_{x\to c}F(x)=L$ says: for every $\varepsilon>0$ there is $\delta>0$ with $|F(x)-L|<\varepsilon$ whenever $x\in A$ and $0<|x-c|<\delta$. Under $h=x-c$ the condition $0<|x-c|<\delta$ reads $0<|h|<\delta$ and $|F(x)-L| = |G(h)-L|$, which is word for word $\lim_{h\to 0}G(h)=L$. Either limit may therefore serve as the definition of $g'(c)$.
(b) Openness of A gives $\delta_0>0$ with $V_{\delta_0}(c)\subseteq A$, so both $c+h$ and $c-h$ lie in A for $0<|h|<\delta_0$ and the symmetric quotient is defined there. Split it:

$$\begin{aligned}\frac{g(c+h)-g(c-h)}{2h} &= \frac{\big(g(c+h)-g(c)\big)+\big(g(c)-g(c-h)\big)}{2h}\\ &= \frac{1}{2}\cdot\frac{g(c+h)-g(c)}{h} + \frac{1}{2}\cdot\frac{g(c-h)-g(c)}{-h}.\end{aligned}$$

By part (a) the first quotient tends to $g'(c)$ as $h\to 0$. For the second, put $k=-h$: as $h\to 0$ we have $k\to 0$, and the quotient is $\big(g(c+k)-g(c)\big)/k$, which tends to $g'(c)$ as well by part (a): any δ that works for the variable k works for h, since $|k|=|h|$. Corollary 4.2.4 now gives

$$\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h} = \tfrac{1}{2}g'(c)+\tfrac{1}{2}g'(c) = g'(c).$$

Problem 5.2.7. Let

$$g_a(x) = \begin{cases} x^a\sin(1/x) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Find a particular (potentially noninteger) value for a so that
(a) g_a is differentiable on $\mathbf{R}$ but such that g_a' is unbounded on $[0,1]$.
(b) g_a is differentiable on $\mathbf{R}$ with g_a' continuous but not differentiable at zero.
(c) g_a is differentiable on $\mathbf{R}$ and g_a' is differentiable on $\mathbf{R}$, but such that g_a'' is not continuous at zero.

Solution. (a) $a = 3/2$; (b) $a = 5/2$; (c) $a = 7/2$. For noninteger a read x^a as $|x|^a$, so that g_a is defined on all of $\mathbf{R}$; every estimate below bounds $|g_a'|$ or $|g_a''|$ by powers of $|x|$ and so is valid for both signs, and every witnessing sequence below is positive.
Two computations govern all three parts. First, for $x > 0$ the Chain and Product Rules (Theorems 5.2.5, 5.2.4 (iii)) give

$$\begin{aligned} g_a'(x) &= ax^{a-1}\sin(1/x) - x^{a-2}\cos(1/x), \\ g_a''(x) &= a(a-1)x^{a-2}\sin(1/x) \\ &\quad - (2a-2)x^{a-3}\cos(1/x) - x^{a-4}\sin(1/x). \end{aligned}$$

Second, at the origin the difference quotient of g_a is

$$\frac{g_a(x) - g_a(0)}{x - 0} = x^{a-1}\sin(1/x),$$

which is bounded by $|x|^{a-1}$ and hence tends to 0 when $a > 1$; when $a \le 1$ it fails to converge, since along $x_n = 1/(2\pi n + \pi/2)$ it equals $x_n^{a-1} \not\to 0$ while along $x_n = 1/(2\pi n)$ it is 0. So

$$g_a \text{ is differentiable on } \mathbf{R} \iff a > 1, \quad \text{and then } g_a'(0) = 0.$$

(a) Take $a = 3/2 > 1$, so g_a is differentiable on $\mathbf{R}$. Along $x_n = 1/(2\pi n) \in [0,1]$ we have $\sin(1/x_n) = 0$ and $\cos(1/x_n) = 1$, so

$$g_{3/2}'(x_n) = -x_n^{-1/2} = -\sqrt{2\pi n} \longrightarrow -\infty.$$

Hence $g_{3/2}'$ is unbounded on $[0,1]$.
(b) Take $a = 5/2$. Since $a > 1$, g_a is differentiable on $\mathbf{R}$ with $g_a'(0) = 0$, and for $x \neq 0$

$$|g_{5/2}'(x)| \le \tfrac{5}{2}|x|^{3/2} + |x|^{1/2} \longrightarrow 0 = g_{5/2}'(0),$$

so $g_{5/2}'$ is continuous at zero, and it is continuous elsewhere by the usual rules. But its difference quotient at zero, for $x > 0$,

$$\frac{g_{5/2}'(x) - g_{5/2}'(0)}{x - 0} = \tfrac{5}{2}x^{1/2}\sin(1/x) - x^{-1/2}\cos(1/x),$$

is unbounded along $x_n = 1/(2\pi n)$ (the first term vanishes there and the second equals $-\sqrt{2\pi n}$), so the limit does not exist and $g_{5/2}'$ is not differentiable at zero.
(c) Take $a = 7/2$. Since $a > 1$, g_a is differentiable on $\mathbf{R}$, and its derivative is differentiable at zero because

$$\left|\frac{g_{7/2}'(x) - g_{7/2}'(0)}{x - 0}\right| \le \tfrac{7}{2}|x|^{3/2} + |x|^{1/2} \longrightarrow 0,$$

thus $g_{7/2}''(0) = 0$, and $g_{7/2}''$ exists away from zero by the formula above. However, along $x_n = 1/(2\pi n + \pi/2)$, where $\sin(1/x_n) = 1$ and $\cos(1/x_n) = 0$,

$$g_{7/2}''(x_n) = \tfrac{35}{4}x_n^{3/2} - x_n^{-1/2} \longrightarrow -\infty,$$

while $g_{7/2}''(0) = 0$. So $g_{7/2}''$ is not continuous at zero.

5.2 Exercises 5.2.8–5.3.2

Problem 5.2.8. Review the definition of uniform continuity (Definition 4.4.4). Given a differentiable function $f : A \to \mathbf{R}$, let us say that f is *uniformly differentiable* on A if, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \epsilon \quad \text{whenever } 0 < |x - y| < \delta.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbf{R}$? How about $g(x) = x^3$?
(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.
(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Solution. (a) Yes for $f(x) = x^2$, no for $g(x) = x^3$. For $x \neq y$,

$$\frac{x^2 - y^2}{x - y} - 2y = (x + y) - 2y = x - y,$$

so $\delta = \epsilon$ works. For the cube,

$$\frac{x^3 - y^3}{x - y} - 3y^2 = x^2 + xy - 2y^2 = (x - y)(x + 2y),$$

which is unbounded once $x - y$ is pinned down. Take $\epsilon = 1$: given any $\delta > 0$, set $x - y = \delta/2$ and let $y = 2/\delta$; then

$$|(x - y)(x + 2y)| = \frac{\delta}{2}\left|\frac{\delta}{2} + \frac{6}{\delta}\right| > 3 > 1,$$

while $0 < |x - y| < \delta$. So g fails the definition on $\mathbf{R}$.
(b) Uniform differentiability on A makes f' *uniformly* continuous on A, hence continuous. Given $\epsilon > 0$, choose $\delta > 0$ for the value $\epsilon/2$ in the definition. If $x, y \in A$ satisfy $0 < |x - y| < \delta$, then both orderings of the pair may be fed into the definition, and since the difference quotient is symmetric,

$$\frac{f(x) - f(y)}{x - y} = \frac{f(y) - f(x)}{y - x},$$

the triangle inequality gives

$$\begin{aligned} |f'(x) - f'(y)| &\leq \left|f'(x) - \frac{f(y) - f(x)}{y - x}\right| \\ &\quad + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

The case $x = y$ is trivial, so $|x - y| < \delta$ implies $|f'(x) - f'(y)| < \epsilon$ for all $x, y \in A$, which is Definition 4.4.4 for f'.
(c) No, differentiability on a closed interval is not enough; the correct analogue of Theorem 4.4.7 requires a *continuous* derivative. For the negative half, part (b) says a uniformly differentiable function has continuous derivative, and

$$h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $[-1, 1]$ with $h'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ and $h'(0) = 0$ (Exercise 5.2.7), and h' is not continuous at 0 since $h'\big(1/(2\pi n)\big) = -1$ for all n. So h is not uniformly differentiable on $[-1, 1]$.
For the positive half: if f is differentiable on $[a, b]$ *and* f' is continuous there, then f is uniformly differentiable on $[a, b]$. Indeed $[a, b]$ is compact, so Theorem 4.4.7 makes f' uniformly continuous: given $\epsilon > 0$ there is $\delta > 0$ with $|f'(s) - f'(t)| < \epsilon$ whenever $|s - t| < \delta$. For $x \neq y$ in $[a, b]$ the Mean Value Theorem (Theorem 5.3.2, applicable since f is continuous on the closed interval with endpoints x, y and differentiable on its interior) supplies a c strictly between x and y with

$$\frac{f(x) - f(y)}{x - y} = f'(c), \qquad |c - y| < |x - y|,$$

so $0 < |x - y| < \delta$ forces $|f'(c) - f'(y)| < \epsilon$, which is exactly the required estimate.

Problem 5.2.9. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.
(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.
(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.
(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution. (a) True; (b) false; (c) true.
(a) Since f' is not constant there are points $a < b$ of the interval with $f'(a) \neq f'(b)$, and $[a, b]$ lies inside the interval. Corollary 1.4.4 supplies an irrational α strictly between $f'(a)$ and $f'(b)$; whichever of the two orderings holds, it is one of the two hypotheses $f'(a) < \alpha < f'(b)$, $f'(a) > \alpha > f'(b)$ of Darboux's Theorem (Theorem 5.2.7), which applies because f is differentiable on $[a, b]$ and produces $c \in (a, b)$ with

$$f'(c) = \alpha \notin \mathbf{Q}.$$

(b) The counterexample is the function of Exercise 5.2.10,

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

which is differentiable on $\mathbf{R}$ with $g'(0) = 1/2 > 0$ and, for $x \neq 0$,

$$g'(x) = \tfrac{1}{2} + 2x \sin(1/x) - \cos(1/x).$$

Every neighborhood $V_\delta(0)$ contains $x_n = 1/(2\pi n)$ for all large n, and there

$$g'(x_n) = \tfrac{1}{2} + 0 - 1 = -\tfrac{1}{2} < 0.$$

So no δ-neighborhood of $c = 0$ has $g' > 0$ throughout.
(c) Fix $h \neq 0$ small enough that $[0, h]$ (or $[h, 0]$) lies in the interval. The Mean Value Theorem (Theorem 5.3.2; f is differentiable, hence continuous by Theorem 5.2.3, on that closed interval) gives a point c_h strictly between 0 and h with

$$\frac{f(h) - f(0)}{h} = f'(c_h), \qquad 0 < |c_h| < |h|.$$

Let (h_n) be any sequence tending to 0 with $h_n \neq 0$. Then $c_{h_n} \to 0$ with $c_{h_n} \neq 0$, so by the Sequential Criterion for Functional Limits (Theorem 4.2.3) applied to f',

$$\frac{f(h_n) - f(0)}{h_n} = f'(c_{h_n}) \longrightarrow L.$$

Theorem 4.2.3 in the other direction now gives $f'(0) = \lim_{h\to 0}(f(h) - f(0))/h = L$.

Problem 5.2.10. Recall that a function $f : (a, b) \to \mathbf{R}$ is *increasing* on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate. Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbf{R}$ and satisfies $g'(0) > 0$. Now, prove that g is *not* increasing over any open interval containing 0.
In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all

$x \in (a, b)$.

Solution. $g'(0) = 1/2$, yet $g' = -1/2$ at every $x_n = 1/(2\pi n)$, and a point of negative derivative kills monotonicity nearby.
Away from the origin g is a composition and product of functions differentiable on $\mathbf{R} \setminus \{0\}$, so by the Algebraic Differentiability Theorem (Theorem 5.2.4) and the Chain Rule (Theorem 5.2.5),

$$g'(x) = \tfrac{1}{2} + 2x\sin(1/x) - \cos(1/x) \qquad (x \neq 0).$$

At the origin, Definition 5.2.1 and $|h\sin(1/h)| \leq |h|$ give

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \left(\tfrac{1}{2} + h\sin(1/h)\right) = \tfrac{1}{2} > 0.$$

So g is differentiable on all of $\mathbf{R}$.
Now let I be any open interval containing 0, and choose $c > 0$ with $(-c, c) \subseteq I$. Fix $n \in \mathbf{N}$ large enough that $x_n = 1/(2\pi n) < c/2$. Since $\sin(2\pi n) = 0$ and $\cos(2\pi n) = 1$,

$$g'(x_n) = \tfrac{1}{2} + 0 - 1 = -\tfrac{1}{2}.$$

Apply Definition 5.2.1 at x_n with $\epsilon = 1/4$: there is $\delta > 0$, which we may shrink so that $x_n + \delta < c$, such that

$$\left|\frac{g(t) - g(x_n)}{t - x_n} + \frac{1}{2}\right| < \frac{1}{4} \quad \text{whenever } 0 < |t - x_n| < \delta.$$

Take $t = x_n + \delta/2$. Then $x_n < t < c$, so both points lie in I; the quotient is $< -1/4 < 0$ while $t - x_n > 0$, so

$$g(t) - g(x_n) < 0, \qquad \text{i.e. } g(t) < g(x_n) \text{ with } x_n < t.$$

This violates the definition of increasing on I.

Problem 5.2.11. Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.
(a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.
(b) Now complete the proof of Darboux's Theorem started earlier. (Theorem 5.2.7: if f is differentiable on an interval $[a, b]$ and α satisfies $f'(a) < \alpha < f'(b)$, or $f'(a) > \alpha > f'(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$. The text reduces this to the case above by setting $g(x) = f(x) - \alpha x$, so that $g'(a) < 0 < g'(b)$, and it remains to produce $c \in (a, b)$ with $g'(c) = 0$.)

Solution. (a) Both points come straight out of Definition 5.2.1 with ϵ half the relevant slope.
At a: apply the definition of $g'(a)$ with $\epsilon = -g'(a)/2 > 0$ to get $\delta > 0$ such that

$$\left|\frac{g(t) - g(a)}{t - a} - g'(a)\right| < \frac{-g'(a)}{2} \quad \text{for } t \in [a, b],\ 0 < |t - a| < \delta,$$

so the quotient is $< g'(a)/2 < 0$ there. Choose any x with $a < x < \min\{a + \delta, b\}$; since $x - a > 0$, the displayed sign forces

$$g(x) - g(a) < 0, \qquad \text{i.e. } g(a) > g(x).$$

At b: apply the definition of $g'(b)$ with $\epsilon = g'(b)/2 > 0$ to get $\delta' > 0$ with the quotient $> g'(b)/2 > 0$ for $t \in [a, b]$, $0 < |t - b| < \delta'$. Choose y with $\max\{a, b - \delta'\} < y < b$. Now $y - b < 0$, so

$$g(y) - g(b) = \underbrace{\frac{g(y) - g(b)}{y - b}}_{>0} \cdot \underbrace{(y - b)}_{<0} < 0,$$

i.e. $g(y) < g(b)$.

(b) Continue with $g(x) = f(x) - \alpha x$, which is differentiable on $[a, b]$ with $g' = f' - \alpha$ and $g'(a) < 0 < g'(b)$. Differentiability implies continuity (Theorem 5.2.3), so g is continuous on the compact set $[a, b]$ and the Extreme Value Theorem (Theorem 4.4.2) gives a point $c \in [a, b]$ with

$$g(c) \leq g(t) \qquad \text{for all } t \in [a, b].$$

Part (a) rules out both endpoints: $g(x) < g(a)$ for some $x \in (a, b)$ shows $c \neq a$, and $g(y) < g(b)$ for some $y \in (a, b)$ shows $c \neq b$. Hence $c \in (a, b)$, and $g(c)$ is in particular a minimum value of g on the open interval (a, b), so the Interior Extremum Theorem (Theorem 5.2.6, whose hypotheses hold since g is differentiable on (a, b)) yields

$$0 = g'(c) = f'(c) - \alpha, \qquad f'(c) = \alpha.$$

For the remaining case $f'(a) > \alpha > f'(b)$, apply what was just proved to $-f$ and $-\alpha$: $(-f)'(a) = -f'(a) < -\alpha < -f'(b) = (-f)'(b)$, so some $c \in (a, b)$ has $-f'(c) = -\alpha$.

Problem 5.2.12. (Inverse functions). If $f : [a, b] \to \mathbf{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let us add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show f^{-1} is differentiable with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \qquad \text{where } y = f(x).$$

Solution. Write $g = f^{-1}$. The whole proof is the algebraic identity

$$\frac{g(y) - g(y_0)}{y - y_0} = \left(\frac{f(x) - f(x_0)}{x - x_0}\right)^{-1}, \qquad x = g(y),\ x_0 = g(y_0),$$

run through the Sequential Criterion for Functional Limits.

The domain of g is $f([a, b]) = [c, d]$ with $c < d$: differentiability gives continuity (Theorem 5.2.3), the Extreme Value Theorem (Theorem 4.4.2) and the Intermediate Value Theorem (Theorem 4.5.1) make the range a closed interval, and $c \neq d$ since f is one-to-one on $[a, b]$. So every point of $[c, d]$ is a limit point of it, and g is continuous there by Exercise 4.5.8.

Fix $y_0 \in [c, d]$ and put $x_0 = g(y_0)$, so $y_0 = f(x_0)$. Let (y_n) be any sequence in $[c, d] \setminus \{y_0\}$ with $y_n \to y_0$, and set $x_n = g(y_n)$. Then:

(i) $x_n \neq x_0$, since g is one-to-one (f is a function) and $y_n \neq y_0$;

(ii) $x_n \to x_0$, by continuity of g at y_0 and Theorem 4.3.2 (iii).

Because f is differentiable at x_0, Theorem 4.2.3 applied to the difference-quotient function $h(t) = (f(t) - f(x_0))/(t - x_0)$ along the sequence (x_n) gives

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \longrightarrow f'(x_0) \neq 0.$$

Since $f(x_n) = y_n$ and $f(x_0) = y_0$, the reciprocal of this quotient is exactly $(g(y_n) - g(y_0))/(y_n - y_0)$, so the Algebraic Limit Theorem (Theorem 2.3.3 (iv), whose hypothesis $f'(x_0) \neq 0$ is our standing assumption) yields

$$\frac{g(y_n) - g(y_0)}{y_n - y_0} = \left(\frac{f(x_n) - f(x_0)}{x_n - x_0}\right)^{-1} \longrightarrow \frac{1}{f'(x_0)}.$$

The sequence (y_n) was arbitrary, so Theorem 4.2.3 in the converse direction gives

$$\left(f^{-1}\right)'(y_0) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)},$$

which is the claimed formula with $y_0 = f(x_0)$.

Problem 5.3.1. Recall from Exercise 4.4.9 that a function $f : A \to \mathbf{R}$ is *Lipschitz* on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le M$$

for all $x \ne y$ in A.
(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.
(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution. (a) Take $M = \sup_{t\in[a,b]} |f'(t)|$, which is finite and attained because $|f'|$ is continuous on the compact set $[a, b]$ (Extreme Value Theorem, Theorem 4.4.2); if this supremum is 0 replace it by $M = 1$ so that $M > 0$. For $x \ne y$ in $[a, b]$, f is continuous on the closed interval between them and differentiable on its interior, so the Mean Value Theorem (Theorem 5.3.2) supplies a c strictly between x and y with

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \le M.$$

(b) Yes, with (a)'s standing hypothesis that f' is continuous. That hypothesis turns the bound into a maximum rather than a mere supremum: $|f'|$ is continuous on the compact set $[a, b]$, so by the Extreme Value Theorem (Theorem 4.4.2) it is attained at some $x_0 \in [a, b]$, whence

$$k := \max_{t\in[a,b]} |f'(t)| = |f'(x_0)| < 1.$$

If $k = 0$ then f is constant and any constant in $(0, 1)$ serves; otherwise $0 < k < 1$, and the display in (a) run with k in place of M gives $|f(x) - f(y)| \le k|x - y|$ for all $x, y \in [a, b]$, the contractive condition of Exercise 4.3.11.

Problem 5.3.2. Let f be differentiable on an interval A. If $f'(x) \ne 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution. Rolle's Theorem, contrapositively. Suppose $x < y$ in A with $f(x) = f(y)$. Since A is an interval, $[x, y] \subseteq A$, so f is differentiable — hence continuous (Theorem 5.2.3) — on $[x, y]$ and differentiable on (x, y). Rolle's Theorem (Theorem 5.3.1) produces $c \in (x, y) \subseteq A$ with

$$f'(c) = 0,$$

contradicting the hypothesis. Thus $f(x) \ne f(y)$ whenever $x \ne y$.
The converse fails: $f(x) = x^3$ on $A = \mathbf{R}$ is strictly increasing, hence one-to-one, but $f'(0) = 0$.

5.3 Exercises 5.3.3–5.3.9

Problem 5.3.3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.
(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.
(b) Argue that at some point c we have $h'(c) = 1/3$.
(c) Argue that $h'(x) = 1/4$ at some point in the domain.

Solution. (a) Apply the Intermediate Value Theorem to $g(x) = h(x) - x$, which is continuous on $[1, 3]$ because h is differentiable there (Theorem 5.2.3). Its endpoint values straddle zero:

$$g(1) = 2 - 1 = 1 > 0, \qquad g(3) = 2 - 3 = -1 < 0,$$

so Theorem 4.5.1 gives $d \in (1, 3) \subseteq [0, 3]$ with $g(d) = 0$, i.e. $h(d) = d$.

(b) The Mean Value Theorem (Theorem 5.3.2) on $[0,3]$ — where h is continuous and differentiable on $(0,3)$ — gives $c \in (0,3)$ with

$$h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{2-1}{3} = \frac{1}{3}.$$

(c) Produce a second derivative value and interpolate. The Mean Value Theorem on $[1,3]$ gives $c_2 \in (1,3)$ with

$$h'(c_2) = \frac{h(3)-h(1)}{3-1} = \frac{2-2}{2} = 0.$$

Together with the point c of (b) we have two values of h' with

$$h'(c_2) = 0 < \tfrac{1}{4} < \tfrac{1}{3} = h'(c).$$

Since h is differentiable on the closed interval with endpoints c and c_2, Darboux's Theorem (Theorem 5.2.7) — derivatives have the intermediate value property, no continuity of h' required — yields a point x strictly between c_2 and c with $h'(x) = 1/4$.

Problem 5.3.4. Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.
(a) If $f(x_n) = 0$ for all $n \in \mathbf{N}$, show $f(0) = 0$ and $f'(0) = 0$.
(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Solution. (a) Feed the sequence (x_n) into the Sequential Criterion for Functional Limits (Theorem 4.2.3) twice. First, f is continuous at 0 (Theorem 5.2.3) and $x_n \to 0$, so

$$f(0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0.$$

Second, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists by hypothesis, and 0 is a limit point of $A \setminus \{0\}$ approached by the x_n (each $x_n \neq 0$), so Theorem 4.2.3 lets us evaluate that limit along (x_n):

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n} = \lim_{n\to\infty} \frac{0-0}{x_n} = 0.$$

(b) The same argument applied to f', once we manufacture zeros of f' accumulating at 0. Since $x_n \to 0$ with every $x_n \neq 0$, infinitely many terms share a sign; say infinitely many are positive (the negative case is identical with the inequalities reversed). Choose a subsequence strictly decreasing to 0,

$$x_{n_1} > x_{n_2} > x_{n_3} > \cdots > 0, \qquad x_{n_k} \to 0,$$

which is possible because for each k there are infinitely many indices n with $0 < x_n < x_{n_k}$. On $[x_{n_{k+1}}, x_{n_k}] \subseteq A$ the function f is continuous and differentiable with

$$f(x_{n_{k+1}}) = 0 = f(x_{n_k}),$$

so Rolle's Theorem (Theorem 5.3.1) supplies $y_k \in (x_{n_{k+1}}, x_{n_k})$ with $f'(y_k) = 0$. Then $0 < y_k < x_{n_k} \to 0$ forces $y_k \to 0$ with $y_k \neq 0$. By (a), $f'(0) = 0$; and since $f''(0) = \lim_{y\to 0} \frac{f'(y)-f'(0)}{y}$ exists, Theorem 4.2.3 evaluates it along (y_k):

$$f''(0) = \lim_{k\to\infty} \frac{f'(y_k)-f'(0)}{y_k} = \lim_{k\to\infty} \frac{0-0}{y_k} = 0.$$

Problem 5.3.5. (a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem (Theorem 5.3.5): if f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve.)

Solution. (a) Apply the Mean Value Theorem to the auxiliary function named in the book's proof,

$$h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x).$$

As a linear combination of f and g, h is continuous on $[a, b]$ and differentiable on (a, b) (Algebraic Differentiability Theorem, Theorem 5.2.4), and the cross terms cancel at the endpoints:

$$\begin{aligned} h(a) &= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) = f(b)g(a) - g(b)f(a), \\ h(b) &= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) = g(a)f(b) - f(a)g(b). \end{aligned}$$

Since $h(a) = h(b)$, the Mean Value Theorem (Theorem 5.3.2) gives $c \in (a, b)$ with $h'(c) = (h(b) - h(a))/(b - a) = 0$, i.e.

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0,$$

which is the asserted identity. If in addition $g' \neq 0$ on (a, b), then $g(b) \neq g(a)$ — otherwise Rolle's Theorem (Theorem 5.3.1) applied to g alone would produce a zero of g' — so dividing the identity by $g'(c)[g(b) - g(a)]$, both factors nonzero, gives

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(b) Read $t \mapsto (g(t), f(t))$, $t \in [a, b]$, as a parametrized curve in the plane, with g the horizontal and f the vertical coordinate. The chord joining its endpoints $(g(a), f(a))$ and $(g(b), f(b))$ has slope

$$\frac{f(b) - f(a)}{g(b) - g(a)},$$

while the tangent vector at parameter c is $(g'(c), f'(c))$, of slope $f'(c)/g'(c)$. The theorem says some tangent to the curve is parallel to that chord — the picture of Figure 5.4, now for a curve traced parametrically rather than a graph — and the unquotiented form $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ records that parallelism as a vanishing cross product, valid even where a slope is infinite.

Problem 5.3.6. (a) Let $g : [0, a] \to \mathbf{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.
(b) Let $h : [0, a] \to \mathbf{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.
(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Solution. (a) Fix $x \in (0, a]$ (the case $x = 0$ is trivial) and apply the Mean Value Theorem (Theorem 5.3.2) to g on $[0, x]$, where g is continuous and differentiable on $(0, x)$: there is $c \in (0, x)$ with

$$|g(x)| = |g(x) - g(0)| = |g'(c)|\, x \leq Mx.$$

(b) Compare h against $x^2/2$ using the Generalized Mean Value Theorem rather than the plain one, which is what buys the factor 1/2. Part (a) applied to $g = h'$ (differentiable, $h'(0) = 0$, $|(h')'| = |h''| \le M$) gives

$$|h'(t)| \le Mt \qquad \text{for all } t \in [0, a].$$

Fix $x \in (0, a]$ and set $G(t) = t^2/2$, so $G'(t) = t \neq 0$ on $(0, x)$. Both h and G are continuous on $[0, x]$ and differentiable on $(0, x)$, so Theorem 5.3.5 yields $c \in (0, x)$ with

$$\frac{h(x)}{x^2/2} = \frac{h(x) - h(0)}{G(x) - G(0)} = \frac{h'(c)}{G'(c)} = \frac{h'(c)}{c},$$

and the displayed bound on $|h'|$ makes $|h'(c)|/c \le M$. Hence $|h(x)| \le Mx^2/2$.
(c) Conjecture: if $k : [0, a] \to \mathbf{R}$ is three times differentiable with

$$k(0) = k'(0) = k''(0) = 0 \qquad \text{and} \qquad |k'''(x)| \le M \ \text{ on } [0, a],$$

then $|k(x)| \le Mx^3/6$ for all $x \in [0, a]$.
Proof: part (b) applied to $h = k'$ (twice differentiable, $k'(0) = (k')'(0) = k''(0) = 0$, $|(k')''| = |k'''| \le M$) gives

$$|k'(t)| \le \frac{Mt^2}{2} \qquad \text{for all } t \in [0, a].$$

Fix $x \in (0, a]$ and set $G(t) = t^3/6$, so $G'(t) = t^2/2 \neq 0$ on $(0, x)$. Theorem 5.3.5 on $[0, x]$ (again both functions continuous there, differentiable inside) gives $c \in (0, x)$ with

$$\frac{k(x)}{x^3/6} = \frac{k(x) - k(0)}{G(x) - G(0)} = \frac{k'(c)}{c^2/2},$$

whose modulus is at most M by the previous display. Hence $|k(x)| \le Mx^3/6$.

Problem 5.3.7. A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Solution. Two fixed points would force the chord slope to be 1. Suppose $x < y$ are both fixed points of f in the interval A. Since A is an interval, $[x, y] \subseteq A$, so f is continuous on $[x, y]$ (Theorem 5.2.3) and differentiable on (x, y), and the Mean Value Theorem (Theorem 5.3.2) gives $c \in (x, y)$ with

$$f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1,$$

contradicting $f'(c) \neq 1$. Hence f has at most one fixed point.
Method (2): apply Exercise 5.3.2 to $g(x) = f(x) - x$, whose derivative $g'(x) = f'(x) - 1$ is never zero; g is therefore one-to-one, so $g(x) = 0$ for at most one x.

Problem 5.3.8. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x \to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution. Every difference quotient at 0 equals a value of f' nearby, so it inherits the limit L.
Let $\epsilon > 0$, and choose $\delta > 0$ with $(-\delta, \delta)$ inside the domain and

$$0 < |t| < \delta \quad \Longrightarrow \quad |f'(t) - L| < \epsilon.$$

Fix x with $0 < |x| < \delta$. On the closed interval with endpoints 0 and x the function f is continuous, and it is differentiable on the corresponding open interval, which omits 0; so the Mean Value Theorem (Theorem 5.3.2) supplies a point c_x strictly between 0 and x with

$$\frac{f(x) - f(0)}{x - 0} = f'(c_x), \qquad 0 < |c_x| < |x| < \delta,$$

and therefore

$$\left|\frac{f(x)-f(0)}{x-0}-L\right| = |f'(c_x)-L| < \epsilon.$$

Hence $f'(0)$ exists and equals L.

Problem 5.3.9. Assume f and g are as described in Theorem 5.3.6 (both continuous on an interval containing a, both differentiable on this interval with the possible exception of the point a, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$), but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the 0/0 case of L'Hospital's Rule under this stronger hypothesis.

Solution. Divide the two difference quotients at a and quote the Algebraic Limit Theorem; no Mean Value Theorem is needed.
The standing hypothesis $g'(x) \neq 0$ for $x \neq a$ already forces $g(x) \neq 0$ for $x \neq a$ (else Rolle's Theorem, Theorem 5.3.1, applied between a and x would zero g' somewhere strictly between), so f/g is defined near a. There, using $f(a) = g(a) = 0$,

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f(x)-f(a)}{g(x)-g(a)}\\ &= \frac{\big(f(x)-f(a)\big)/(x-a)}{\big(g(x)-g(a)\big)/(x-a)}.\end{aligned}$$

Numerator and denominator tend to $f'(a)$ and $g'(a)$, and $g'(a) \neq 0$ is exactly the nonvanishing hypothesis of the quotient part of the Algebraic Limit Theorem for Functional Limits (Corollary 4.2.4 (iv)), so

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Since f' and g' are continuous at a with $g'(a) \neq 0$, the same corollary gives

$$L = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)},$$

and the two displays agree.

5.4 Exercises 5.3.10–5.4.4

Problem 5.3.10. Let $f(x) = x\sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of Theorem 5.3.6.

Solution. The first three limits are all 0; the fourth does not exist (finite or infinite).
Both functions are defined for $x \neq 0$ and extended continuously by the value 0 at $x = 0$.
(i) $\lim_{x\to 0} g(x) = 0$, since $1/x^2 \to \infty$ and $e^{-t} \to 0$ as $t \to \infty$.
(ii) $\lim_{x\to 0} f(x) = 0$, by the squeeze $|f(x)| \leq |x|e^{-1/x^2} \leq |x|$.
(iii) For $x \neq 0$ the exponentials cancel:

$$\frac{f(x)}{g(x)} = x\sin(1/x^4), \qquad \big|x\sin(1/x^4)\big| \leq |x|,$$

so $\lim_{x\to 0} f(x)/g(x) = 0$.
(iv) Differentiating,

$$g'(x) = \frac{2}{x^3}e^{-1/x^2},$$

$$f'(x) = e^{-1/x^2}\Big[\sin\frac{1}{x^4} - \frac{4}{x^4}\cos\frac{1}{x^4} + \frac{2}{x^2}\sin\frac{1}{x^4}\Big],$$

and dividing (the exponentials cancel again),

$$\frac{f'(x)}{g'(x)} = \frac{x^3}{2}\sin\frac{1}{x^4} + x\sin\frac{1}{x^4} - \frac{2}{x}\cos\frac{1}{x^4}.$$

The first two terms tend to 0; the third oscillates unboundedly. Take the null sequences $x_n = (2\pi n)^{-1/4}$, where $\sin(1/x_n^4) = 0$ and $\cos(1/x_n^4) = 1$, and $y_n = (\pi/2 + 2\pi n)^{-1/4}$, where the two values are reversed ($n \geq 1$):

$$\frac{f'(x_n)}{g'(x_n)} = -\frac{2}{x_n}, \qquad 0 < \frac{f'(y_n)}{g'(y_n)} = \frac{y_n^3}{2} + y_n < 1.$$

No finite limit L is possible: by the Sequential Criterion for Functional Limits (Theorem 4.2.3) it would force $-2/x_n \to L$, and that sequence is unbounded. Nor is the limit ∞ or $-\infty$, since every punctured neighborhood of 0 contains a point y_n where the quotient lies strictly between 0 and 1, defeating Definition 5.3.7 (and its $-\infty$ analogue) at $M = 1$.
The pair satisfies every hypothesis of Theorem 5.3.6 at $a = 0$: both continuous on an interval containing 0, both differentiable off 0, $f(0) = g(0) = 0$, and $g'(x) = (2/x^3)e^{-1/x^2} \neq 0$ for $x \neq 0$. The surprise is that f/g converges while f'/g' does not, so L'Hospital's Rule is useless here and the divergence of f'/g' says nothing about f/g. There is no conflict: Theorem 5.3.6 is a one-way implication whose hypothesis fails in this example, and its converse is false.

Problem 5.3.11. (a) Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's Rule (Theorem 5.3.6).
(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same but we assume that

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \infty,$$

does it necessarily follow that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \infty?$$

Solution. The Generalized Mean Value Theorem turns $f(x)/g(x)$ into a value of f'/g' nearby; in (b) the answer is yes, by the same identity with M in place of $L \pm \epsilon$.
(a) Let f, g be as in Theorem 5.3.6 on an interval I containing a, and assume $\lim_{x\to a} f'(x)/g'(x) = L$. First, $g(x) \neq 0$ for every $x \in I$ with $x \neq a$. Otherwise $g(a) = g(x) = 0$, and Rolle's Theorem (Theorem 5.3.1) on the closed interval with endpoints a and x (there g is continuous, and differentiable on the interior, which omits a) would give c strictly between them with $g'(c) = 0$, contradicting the hypothesis. So f/g is defined on $I \setminus \{a\}$.
Fix such an x. Both f and g are continuous on the closed interval with endpoints a and x and differentiable on the corresponding open interval, which omits a, so the Generalized Mean Value Theorem (Theorem 5.3.5) supplies c strictly between a and x with

$$\big[f(x) - f(a)\big]g'(c) = \big[g(x) - g(a)\big]f'(c).$$

Since g' is never zero on that open interval and $g(x) - g(a) = g(x) \neq 0$, this can be divided, and $f(a) = g(a) = 0$ gives

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Let $\epsilon > 0$ and choose $\delta > 0$ so that

$$0 < |t-a| < \delta \quad \Longrightarrow \quad \left|\frac{f'(t)}{g'(t)} - L\right| < \epsilon.$$

If $0 < |x-a| < \delta$, the point c above lies strictly between a and x, so $0 < |c-a| < |x-a| < \delta$ and

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon.$$

Hence $\lim_{x\to a} f(x)/g(x) = L$.
(b) Yes. Keeping the same standing hypotheses, the identity

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}, \qquad 0 < |c-a| < |x-a|,$$

established in (a) makes no reference to L. Given $M > 0$, Definition 5.3.7 applied to f'/g' provides $\delta > 0$ with

$$0 < |t-a| < \delta \quad \Longrightarrow \quad \frac{f'(t)}{g'(t)} \geq M.$$

For any x with $0 < |x-a| < \delta$ the associated c satisfies $0 < |c-a| < \delta$, so $f(x)/g(x) = f'(c)/g'(c) \geq M$. Since $M > 0$ was arbitrary, $\lim_{x\to a} f(x)/g(x) = \infty$.

Problem 5.3.12. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

(Compare this to Exercise 5.2.6(b).)

Solution. One application of L'Hospital's Rule in the variable h turns the quotient into a symmetric difference quotient for f' at a.
Choose $r > 0$ with $(a-r, a+r)$ inside the interval on which f is twice differentiable, and for $h \in (-r, r)$ set

$$F(h) = f(a+h) - 2f(a) + f(a-h), \qquad G(h) = h^2.$$

Then F and G are continuous on $(-r, r)$ and differentiable there, with

$$F(0) = G(0) = 0, \qquad F'(h) = f'(a+h) - f'(a-h), \qquad G'(h) = 2h,$$

and $G'(h) = 2h \neq 0$ for $h \neq 0$: every hypothesis of Theorem 5.3.6 holds at the point 0, so it suffices to evaluate $\lim_{h\to 0} F'(h)/G'(h)$.
Since f' is differentiable at a with derivative $f''(a)$, split the symmetric quotient into two ordinary ones:

$$\begin{aligned}\frac{F'(h)}{G'(h)} &= \frac{f'(a+h) - f'(a-h)}{2h} \\ &= \frac{1}{2}\cdot\frac{f'(a+h) - f'(a)}{h} + \frac{1}{2}\cdot\frac{f'(a-h) - f'(a)}{-h}.\end{aligned}$$

Both increments h and $-h$ tend to 0, so each quotient converges to $f''(a)$ and $\lim_{h\to 0} F'(h)/G'(h) = f''(a)$. By Theorem 5.3.6,

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

Method (2), the route through the continuity of f'': apply Theorem 5.3.6 a second time instead of splitting. The pair F', G' again satisfies $F'(0) = G'(0) = 0$, both are differentiable on $(-r, r)$, and $G''(h) = 2 \neq 0$, while

$$\frac{F''(h)}{G''(h)} = \frac{f''(a+h) + f''(a-h)}{2} \longrightarrow f''(a)$$

because f'' is continuous at a.

Problem 5.4.1. Let $h(x) = |x|$ on $[-1, 1]$, extended to all of $\mathbf{R}$ by the requirement $h(x+2) = h(x)$, so that h is the periodic sawtooth of Figure 5.6. Sketch a graph of $(1/2)h(2x)$ on $[-2, 3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n} h(2^n x)$$

as n gets larger.

Solution. $(1/2)h(2x) = h_1(x)$ is the same sawtooth compressed by a factor of 2 in both directions: it is piecewise linear with slopes ± 1, vanishes at every integer, and rises to the value $1/2$ at every half-integer. On $[-2, 3]$ its corners are

x	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3
$h_1(x)$	0	$1/2$	0	$1/2$	0	$1/2$	0	$1/2$	0	$1/2$	0

with straight segments joining consecutive corners: since h_1 has period 1, that is five complete teeth over the length-5 interval $[-2, 3]$.

In general h_n is the sawtooth scaled by 2^{-n} in both coordinates: it is 2^{1-n}-periodic, vanishes exactly at the dyadic points $k/2^{n-1}$, attains its maximum 2^{-n} exactly at the odd multiples $(2k+1)/2^n$, and is affine with slope ± 1 between consecutive points of $2^{-n}\mathbf{Z}$. So as n grows the teeth get shorter and narrower at the same rate: the amplitude 2^{-n} tends to 0 while the number of teeth per unit length doubles, yet $|h_n'| = 1$ wherever the derivative exists.

Problem 5.4.2. Fix $x \in \mathbf{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

converges and thus $g(x)$ is properly defined.

Solution. Compare with the geometric series. Since h takes values in $[0, 1]$,

$$0 \le \frac{1}{2^n} h(2^n x) \le \frac{1}{2^n} \qquad \text{for every } n \ge 0,$$

and $\sum_{n=0}^{\infty} (1/2)^n$ converges (geometric, ratio $1/2 < 1$; Example 2.7.5). The Comparison Test (Theorem 2.7.4), whose hypothesis $0 \le a_n \le b_n$ is exactly the display above, gives convergence of $\sum_{n=0}^{\infty} 2^{-n} h(2^n x)$, so $g(x)$ is defined for every $x \in \mathbf{R}$.

Problem 5.4.3. Taking the continuity of $h(x)$ as given, reference the proper theorems from Chapter 4 that imply that the finite sum

$$g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x)$$

is continuous on $\mathbf{R}$.

Solution. Composition, then the Algebraic Continuity Theorem, then induction on m.

Fix n and set $\ell_n(x) = 2^n x$. The identity function is continuous on $\mathbf{R}$, so ℓ_n is continuous by Theorem 4.3.4 (ii) (a constant multiple of a continuous function). Since h is continuous at every point of $\mathbf{R}$ – in particular at $\ell_n(c)$ for each c – Theorem 4.3.9 (composition of continuous functions) applies and $h \circ \ell_n$ is continuous on $\mathbf{R}$. Theorem 4.3.4 (ii) again gives continuity of

$$h_n(x) = \frac{1}{2^n} (h \circ \ell_n)(x).$$

Finally $g_0 = h_0$ is continuous, and if g_{m-1} is continuous then $g_m = g_{m-1} + h_m$ is continuous by Theorem 4.3.4 (i) (sums of continuous functions). By induction g_m is continuous on $\mathbf{R}$ for every $m \in \mathbf{N} \cup \{0\}$.

Problem 5.4.4. As the graph in Figure 5.7 suggests, the structure of

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.
(a) How do we know g attains a maximum value M on $[0,2]$? What is this value?
(b) Let D be the set of points in $[0,2]$ where g attains its maximum. That is $D = \{x \in [0,2] : g(x) = M\}$. Find one point in D.
(c) Is D finite, countable, or uncountable?

Solution. (a) $M = 4/3$. Existence is the Extreme Value Theorem (Theorem 4.4.2): g is continuous by assumption and $[0,2]$ is a closed bounded interval, hence compact, so g attains a maximum on it. For the value, note first that $h(t) = \operatorname{dist}(t, 2\mathbf{Z})$, since this distance function is 2-periodic and equals $|t|$ on $[-1,1]$. Pair consecutive terms of the series:

$$g(x) = \sum_{k=0}^{\infty} 4^{-k} \Big[h(y_k) + \tfrac{1}{2} h(2y_k) \Big], \qquad y_k = 4^k x.$$

Lemma: $h(y) + \frac{1}{2}h(2y) \le 1$ for all y, with equality exactly when y lies in $[1/2, 3/2]$ modulo 2. Both sides are 2-periodic, so it suffices to check $y \in [0,2]$:

$$\begin{aligned}
&\text{(i) } y \in [0, \tfrac{1}{2}]: && y + \tfrac{1}{2}(2y) = 2y \le 1,\\
&\text{(ii) } y \in [\tfrac{1}{2}, 1]: && y + \tfrac{1}{2}(2 - 2y) = 1,\\
&\text{(iii) } y \in [1, \tfrac{3}{2}]: && (2-y) + \tfrac{1}{2}(2y - 2) = 1,\\
&\text{(iv) } y \in [\tfrac{3}{2}, 2]: && (2-y) + \tfrac{1}{2}(4 - 2y) = 4 - 2y \le 1.
\end{aligned}$$

In (i) equality forces $y = 1/2$ and in (iv) it forces $y = 3/2$, which proves the equality clause. Summing the lemma against the weights 4^{-k} gives

$$g(x) \;\le\; \sum_{k=0}^{\infty} 4^{-k} \;=\; \frac{4}{3} \qquad \text{for every } x \in \mathbf{R},$$

and the bound is attained at $x = 2/3$: for every $n \ge 0$ one has $2^n \cdot \frac{2}{3} \equiv \frac{2}{3}$ or $\frac{4}{3} \pmod 2$ according as n is even or odd, and $h(2/3) = h(4/3) = 2/3$, so

$$g\big(\tfrac{2}{3}\big) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{2}{3} = \frac{2}{3} \cdot 2 = \frac{4}{3}.$$

(b) $x = 2/3 \in D$, by the computation just made.
(c) D is uncountable. By the equality clause of the lemma, $g(x) = 4/3$ precisely when

$$4^k x \in \big[\tfrac{1}{2}, \tfrac{3}{2}\big] \pmod 2 \qquad \text{for every } k \ge 0.$$

Now build such x in base 4. For a sequence $(d_k)_{k \ge 1}$ with each $d_k \in \{1,2\}$, set

$$x = 2z, \qquad z = \sum_{k=1}^{\infty} \frac{d_k}{4^k}.$$

For each $j \ge 0$, $4^j z$ differs by an integer from $\sum_{k \ge 1} d_{k+j} 4^{-k}$, and this tail lies in $\big[\frac{1}{3}, \frac{2}{3}\big]$ because $\sum_{k \ge 1} 4^{-k} = 1/3$ and the digits are between 1 and 2. Hence $4^j z \equiv t_j \pmod 1$ with $t_j \in [1/3, 2/3] \subset [1/4, 3/4]$, so

$$4^j x \equiv 2t_j \pmod 2, \qquad 2t_j \in \big[\tfrac{2}{3}, \tfrac{4}{3}\big] \subset \big[\tfrac{1}{2}, \tfrac{3}{2}\big],$$

and therefore $g(x) = 4/3$. Also $x = 2z \in [2/3, 4/3] \subset [0, 2]$, so $x \in D$.
Distinct digit sequences give distinct points: if (d_k) and (e_k) first differ at index k, with $d_k > e_k$ say, then

$$z - w \;\geq\; \frac{1}{4^k} - \sum_{j>k} \frac{1}{4^j} \;=\; \frac{1}{4^k} - \frac{1}{3 \cdot 4^k} \;>\; 0.$$

So D contains a one-to-one image of the set of all sequences of 1s and 2s, which is uncountable by Exercise 1.6.4 (relabel $1 \mapsto 0$, $2 \mapsto 1$). Hence D is uncountable.

5.5 Exercises 5.4.5–5.4.8

Problem 5.4.5. Let $x_m = 1/2^m$, where $m = 0, 1, 2, \ldots$. Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$

and use this to prove that $g'(0)$ does not exist.

Solution. Only the first $m+1$ terms of the series survive, and each contributes exactly $1/2^m$. Since $g(0) = 0$,

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h\left(2^{n-m}\right),$$

and the terms split at $n = m$: (i) $0 \leq n \leq m$: here $2^{n-m} \in (0, 1]$, so $h(2^{n-m}) = 2^{n-m}$ and the term equals $2^{-n}2^{n-m} = 2^{-m}$.
(ii) $n > m$: here 2^{n-m} is an even integer, so $h(2^{n-m}) = 0$ and the term vanishes. Adding the $m+1$ surviving terms,

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{(m+1)2^{-m}}{2^{-m}} = m + 1.$$

Now $x_m \to 0$ with $x_m \neq 0$, so if the limit defining

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

(Definition 5.2.1) existed and equalled $L \in \mathbf{R}$, the sequential characterization of functional limits (Theorem 4.2.3) would force $m + 1 \to L$. The sequence $(m+1)$ is unbounded, hence divergent, so no such L exists and g is not differentiable at 0.

Problem 5.4.6. (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.
(b) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Solution. (a) Approach from the right along $x_m = 1 + 1/2^m$; the difference quotients equal $m - 1$.
First $g(1) = h(1) = 1$, since $h(2^n) = 0$ for $n \geq 1$. Next, for $n \geq 1$ the number 2^n is an even integer, so 2-periodicity gives $h(2^n + 2^{n-m}) = h(2^{n-m})$, which equals 2^{n-m} for $1 \leq n \leq m$ and 0 for $n > m$. Since $1 + 2^{-m} \in [1, 2]$ we also have $h(1 + 2^{-m}) = 1 - 2^{-m}$. Hence

$$g\left(1 + \tfrac{1}{2^m}\right) = \left(1 - \tfrac{1}{2^m}\right) + \sum_{n=1}^{m} \frac{1}{2^n} \cdot \frac{2^n}{2^m} = 1 + \frac{m-1}{2^m},$$

so that

$$\frac{g(1 + 2^{-m}) - g(1)}{2^{-m}} = m - 1 \longrightarrow \infty.$$

As in Exercise 5.4.5, Theorem 4.2.3 rules out a finite limit for the difference quotients, so $g'(1)$ does not exist.

At $x = 1/2$: $g(1/2) = h(1/2) + \frac{1}{2}h(1) = \frac{1}{2} + \frac{1}{2} = 1$, the terms with $n \geq 2$ vanishing because 2^{n-1} is then an even integer. Take $m \geq 2$ and $x_m = 1/2 + 1/2^m$. Then $h(1/2 + 2^{-m}) = 1/2 + 2^{-m}$ (the argument lies in $(1/2, 1)$), while $n = 1$ contributes $\frac{1}{2}h(1 + 2^{1-m}) = \frac{1}{2}(1 - 2^{1-m}) = \frac{1}{2} - 2^{-m}$, and each $2 \leq n \leq m$ contributes $2^{-n}h(2^{n-m}) = 2^{-m}$ with the rest zero. Adding,

$$g\big(\tfrac{1}{2} + \tfrac{1}{2^m}\big) = 1 + \frac{m-1}{2^m}, \qquad \frac{g(x_m) - g(1/2)}{x_m - 1/2} = m - 1 \longrightarrow \infty,$$

so $g'(1/2)$ does not exist.
(b) Fix $x = p/2^k$ and again take $x_m = x + 1/2^m$ with $m > k$. Both series converge by Exercise 5.4.2, so Theorem 2.7.1 lets us subtract them termwise; since $2^{-n}/2^{-m} = 1/2^{n-m}$,

$$\frac{g(x_m) - g(x)}{x_m - x} = \sum_{n=0}^{\infty} \frac{h(2^n x + 2^{n-m}) - h(2^n x)}{2^{n-m}},$$

each summand being a difference quotient of h across an interval of length 2^{n-m}. Sort the terms:
(i) $n > m$: 2^{n-m} is an even integer, so by 2-periodicity the numerator is 0.
(ii) $k < n \leq m$: $2^n x = 2^{n-k}p$ is an even integer, so $h(2^n x) = 0$, while $h(2^n x + 2^{n-m}) = h(2^{n-m}) = 2^{n-m}$ because $2^{n-m} \in (0, 1]$. Each such term equals $+1$, and there are $m - k$ of them.
(iii) $0 \leq n \leq k$: since $h(t) = \operatorname{dist}(t, 2\mathbf{Z})$ we have $h(s) \leq |s - t| + h(t)$ for all s, t, so h is 1-Lipschitz and each of these $k + 1$ terms lies in $[-1, 1]$.
Consequently

$$\frac{g(x_m) - g(x)}{x_m - x} \;\geq\; (m - k) - (k + 1) \longrightarrow \infty \qquad (m \to \infty).$$

The difference quotients along $x_m \to x$ with $x_m \neq x$ are unbounded, hence divergent, so Theorem 4.2.3 rules out a finite value for $g'(x)$.

Problem 5.4.7. (a) First prove the following general lemma: Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, show

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Now use this lemma to show that $g'(x)$ does not exist. (Here x is not a dyadic number, and for each $m \in \mathbf{N} \cup \{0\}$ the points $x_m = p_m/2^m$ and $y_m = (p_m + 1)/2^m$ are the adjacent dyadic points with $x_m < x < y_m$, so that $\lim x_m = \lim y_m = x$.)

Solution. (a) The two-sided quotient is a convex combination of the two one-sided ones. Put

$$\lambda_n = \frac{b_n - a}{b_n - a_n} \in (0, 1), \qquad D_n = \frac{f(b_n) - f(a)}{b_n - a}, \qquad E_n = \frac{f(a) - f(a_n)}{a - a_n},$$

all defined because $a_n < a < b_n$ forces $b_n - a > 0$ and $a - a_n > 0$, and because J is open with $a_n, b_n \to a \in J$, so $a_n, b_n \in J$ for all large n (discard the earlier terms). A one-line check gives $1 - \lambda_n = (a - a_n)/(b_n - a_n)$ and

$$\lambda_n D_n + (1 - \lambda_n)E_n = \frac{f(b_n) - f(a)}{b_n - a_n} + \frac{f(a) - f(a_n)}{b_n - a_n} = \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

Since f is differentiable at a and $b_n \to a$ with $b_n \neq a$, and $a_n \to a$ with $a_n \neq a$, the sequential characterization of functional limits (Theorem 4.2.3) gives $D_n \to f'(a)$ and $E_n \to f'(a)$. Using $0 < \lambda_n < 1$,

$$\begin{aligned}\left|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(a)\right| &= \big|\lambda_n\big(D_n - f'(a)\big) + (1 - \lambda_n)\big(E_n - f'(a)\big)\big| \\ &\leq \big|D_n - f'(a)\big| + \big|E_n - f'(a)\big| \longrightarrow 0.\end{aligned}$$

(b) The quotients across $[x_m, y_m]$ are integers whose parity alternates, so they cannot converge.

Write $Q_m = \dfrac{g(y_m) - g(x_m)}{y_m - x_m}$ with $y_m - x_m = 2^{-m}$. Since $h_n(t) = 2^{-n}h(2^n t)$ has corners exactly at the points of $2^{-n}\mathbf{Z}$ and is affine with slope ± 1 between them, and since $2^{-n}\mathbf{Z} \subseteq 2^{-m}\mathbf{Z}$ for $n \le m$ while the open interval (x_m, y_m) contains no point of $2^{-m}\mathbf{Z}$, each h_n with $n \le m$ is affine of slope $\varepsilon_n \in \{-1, +1\}$ on $[x_m, y_m]$. For $n > m$ the increment $2^n(y_m - x_m) = 2^{n-m}$ is an even integer, so $h_n(y_m) = h_n(x_m)$ and that term drops out. Therefore

$$Q_m = \sum_{n=0}^{\infty} \frac{h_n(y_m) - h_n(x_m)}{y_m - x_m} = \sum_{n=0}^{m} \varepsilon_n,$$

a sum of $m+1$ terms each equal to ± 1. (Term-by-term evaluation is legitimate: the series for $g(y_m)$ and $g(x_m)$ both converge by Exercise 5.4.2, so the Algebraic Limit Theorem for Series, Theorem 2.7.1, lets us subtract them termwise.) In particular Q_m is an integer with

$$Q_m \equiv m + 1 \pmod 2,$$

so $|Q_{m+1} - Q_m| \ge 1$ for every m and (Q_m) is not Cauchy; by Theorem 2.6.4 it diverges.
Now suppose g were differentiable at x. Applying part (a) with $f = g$, $J = \mathbf{R}$, $a = x$, $a_m = x_m$ and $b_m = y_m$ – the hypotheses $x_m < x < y_m$ and $\lim x_m = \lim y_m = x$ hold because x is not dyadic and $y_m - x_m = 2^{-m}$ – we would get $Q_m \to g'(x)$, contradicting divergence. Hence $g'(x)$ does not exist.

Problem 5.4.8. Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty}(1/2^n)h(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty}(1/3^n)h(2^n x)$?

Solution. Yes for $\sum_{n=0}^{\infty}(1/2^n)h(3^n x)$ (bracket x by triadic rather than dyadic points); no for $\sum_{n=0}^{\infty}(1/3^n)h(2^n x)$, whose difference quotients all lie in $[-3, 3]$.
Review. For x nondyadic put $x_m = p_m/2^m < x < y_m = (p_m + 1)/2^m$. Two facts drive everything. (i) For $n > m$, $2^n x_m = 2^{n-m}p_m$ and $2^n y_m = 2^{n-m}(p_m + 1)$ are even integers, where h vanishes, so those terms cancel out of the difference. (ii) For $n \le m$ the image interval $[2^n x_m, 2^n y_m]$ joins consecutive multiples of $2^{n-m} \le 1$, hence contains no integer in its interior, so $t \mapsto h(2^n t)$ is affine on $[x_m, y_m]$ with slope $\epsilon_n 2^n$, $\epsilon_n = \pm 1$. Hence

$$\frac{g(y_m) - g(x_m)}{y_m - x_m} = \sum_{n=0}^{m} \epsilon_n,$$

an integer of the same parity as $m+1$. Were $g'(x)$ to exist, Exercise 5.4.7 (a) would force this sequence to converge; a convergent integer sequence is eventually constant, while the parity alternates.
Case $f(x) = \sum_{n=0}^{\infty}(1/2^n)h(3^n x)$ (here $ab = 3/2 > 1$). Take x not a triadic rational and bracket it by $x_m = p_m/3^m < x < y_m = (p_m + 1)/3^m$. Fact (ii) survives verbatim: for $n \le m$, $[3^n x_m, 3^n y_m]$ joins consecutive multiples of $3^{n-m} \le 1$, so $h(3^n t)$ is affine on $[x_m, y_m]$ with slope $\epsilon_n 3^n$, $\epsilon_n = \pm 1$; these intervals are nested, so ϵ_n does not depend on m. Fact (i) fails: 3^{n-m} is odd, so for $n > m$ the integers $3^n x_m$ and $3^n y_m$ have the parities of p_m and $p_m + 1$, giving $h(3^n y_m) - h(3^n x_m) = \sigma_m := (-1)^{p_m}$. Also $\epsilon_m = \sigma_m$, since h increases on $[p_m, p_m + 1]$ exactly when p_m is even. So with $A_m = \sum_{n=0}^{m} \epsilon_n (3/2)^n$ and $A_{-1} = 0$,

$$\begin{aligned} D_m := \frac{f(y_m) - f(x_m)}{y_m - x_m} &= A_m + 3^m \sigma_m \sum_{n=m+1}^{\infty} 2^{-n} \\ &= A_m + \sigma_m (3/2)^m = A_{m-1} + 2\sigma_m (3/2)^m, \end{aligned}$$

the last step because $A_m - A_{m-1} = \epsilon_m (3/2)^m = \sigma_m (3/2)^m$. Consecutive difference quotients therefore satisfy, for $m \ge 1$,

$$D_m - D_{m-1} = 2\sigma_m (3/2)^m - \sigma_{m-1}(3/2)^{m-1},$$

so $|D_m - D_{m-1}| \geq 2(3/2)^m - (3/2)^{m-1} = 2(3/2)^{m-1} \to \infty$. But if $f'(x)$ existed then $D_m \to f'(x)$ by Exercise 5.4.7 (a) (its hypotheses hold: $x_m < x < y_m$ and $x_m, y_m \to x$), forcing $D_m - D_{m-1} \to 0$. So $f'(x)$ does not exist.

Case $f(x) = \sum_{n=0}^{\infty}(1/3^n)h(2^n x)$ (here $ab = 2/3 < 1$). The argument collapses. Bracketing a nondyadic x by $x_m = p_m/2^m$ and $y_m = (p_m + 1)/2^m$, facts (i) and (ii) hold exactly as in the review, so

$$\frac{f(y_m) - f(x_m)}{y_m - x_m} = \sum_{n=0}^{m} \epsilon_n (2/3)^n,$$

which converges as $m \to \infty$ by comparison with $\sum(2/3)^n$: no contradiction. Nor can any variant of it succeed, since h is 1-Lipschitz and therefore

$$|f(s) - f(t)| \leq \sum_{n=0}^{\infty} \frac{|h(2^n s) - h(2^n t)|}{3^n} \leq \sum_{n=0}^{\infty} (2/3)^n |s - t| = 3|s - t|,$$

so every difference quotient of f lies in $[-3, 3]$.

6 Sequences and Series of Functions

Contents

6.1 Exercises 6.2.1–6.2.7

Problem 6.2.1. Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$
(a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.
(b) Is the convergence uniform on $(0,\infty)$?
(c) Is the convergence uniform on $(0,1)$?
(d) Is the convergence uniform on $(1,\infty)$?

Solution. (a) $f(x) = 1/x$. Dividing numerator and denominator by n,
$$f_n(x) = \frac{x}{\frac{1}{n}+x^2} \longrightarrow \frac{x}{x^2} = \frac{1}{x}$$
for each fixed $x > 0$, by the Algebraic Limit Theorem (Theorem 2.3.3).
The error, which serves (b)-(d): for $x > 0$,
$$\begin{aligned}\left|f_n(x) - \frac{1}{x}\right| &= \left|\frac{nx^2-(1+nx^2)}{x(1+nx^2)}\right| \\ &= \frac{1}{x(1+nx^2)}.\end{aligned}$$
(b) No. Evaluating at $x = 1/n$ gives
$$|f_n(1/n) - n| = \frac{n}{1+\frac{1}{n}} \geq \frac{n}{2},$$
so $\sup_{x\in(0,\infty)} |f_n(x) - f(x)| = \infty$ for every n; the definition of uniform convergence (Definition 6.2.3) fails for, say, $\epsilon = 1$.
(c) No. The points $x = 1/n$ lie in $(0,1)$, so the computation in (b) applies verbatim.
(d) Yes. For $x > 1$ we have $x \geq 1$ and $1 + nx^2 \geq 1 + n$, hence
$$\left|f_n(x) - \frac{1}{x}\right| = \frac{1}{x(1+nx^2)} \leq \frac{1}{1+n}$$
for all $x \in (1,\infty)$. Given $\epsilon > 0$, choose $N > 1/\epsilon$; then $n \geq N$ forces $|f_n(x) - f(x)| \leq 1/(1+n) < \epsilon$ for every $x \in (1,\infty)$.

Problem 6.2.2. (a) Define a sequence of functions on $\mathbf{R}$ by
$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$
and let f be the pointwise limit of f_n. Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbf{R}$? Is f continuous at zero?
(b) Repeat this exercise using the sequence of functions
$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$
(c) Repeat the exercise once more with the sequence
$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem (Theorem 6.2.6).

Solution. The three answers are (a) yes, no, no; (b) yes, yes, yes; (c) yes, no, yes. Write $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Each of f_n, g_n, h_n vanishes off $\{1, \frac{1}{2}, \ldots, \frac{1}{n}\} \subseteq [1/n, \infty)$, hence is identically zero on $(-1/n, 1/n)$ and continuous at zero – that settles the first question in all three parts.
(a) For fixed x, $f_n(x)$ is eventually constant, giving the pointwise limit

$$f(x) = \begin{cases} 1 & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases}$$

The convergence is not uniform: $f(1/(n+1)) - f_n(1/(n+1)) = 1 - 0 = 1$, so $\sup_{x\in\mathbf{R}} |f_n(x) - f(x)| = 1$ for every n. And f is not continuous at zero, since $f(0) = 0$ while $f(1/k) = 1$ for all k, so the sequence $1/k \to 0$ has $f(1/k) \to 1 \neq f(0)$ (Theorem 4.3.2).
This is consistent with Theorem 6.2.6: the theorem concludes continuity of the limit only under uniform convergence, and here the convergence is merely pointwise.
(b) The pointwise limit is

$$g(x) = \begin{cases} x & \text{if } x \in H \\ 0 & \text{otherwise,} \end{cases}$$

and $g - g_n$ is supported on $\{1/k : k > n\}$, where it equals $1/k$. Hence

$$\sup_{x\in\mathbf{R}} |g_n(x) - g(x)| = \frac{1}{n+1} \longrightarrow 0,$$

so $g_n \to g$ uniformly. Continuity of g at zero now follows from Theorem 6.2.6 (each g_n is continuous at 0), and is directly visible from $|g(x)| \leq |x|$.
(c) For fixed $x = 1/k$ and $n > k$ we have $h_n(1/k) = 1/k$, so the pointwise limit is again $h = g$. The convergence is not uniform:

$$|h_n(1/n) - h(1/n)| = \left|1 - \frac{1}{n}\right| \geq \frac{1}{2}$$

for $n \geq 2$. Nevertheless $h = g$ is continuous at zero.
Part (c) is consistent with Theorem 6.2.6 because that theorem is a one-way implication: uniform convergence is sufficient for the limit to inherit continuity, not necessary.

Problem 6.2.3. For each $n \in \mathbf{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n. \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):
(a) Find the pointwise limit on $[0, \infty)$.
(b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution. (a) The limits are

$$g(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1, \end{cases} \qquad h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

For g: if $0 \leq x < 1$ then $x^n \to 0$ and $g_n(x) \to x/(1+0) = x$; at $x = 1$, $g_n(1) = 1/2$ for every n; if $x > 1$ then $x^n \to \infty$, and $0 < g_n(x) \leq x/x^n = x^{1-n} \to 0$. For h: $h_n(0) = 0$ for all n, while for fixed $x > 0$

every $n > 1/x$ gives $h_n(x) = 1$.
(b) Each g_n is continuous on $[0, \infty)$ (a quotient of continuous functions with $1 + x^n \geq 1 > 0$, Theorem 4.3.4), and each h_n is continuous on $[0, \infty)$ (the two branches agree at the junction: $n \cdot \frac{1}{n} = 1$). But g is discontinuous at $x = 1$ and h is discontinuous at $x = 0$: taking $x_k = 1 + 1/k \to 1$ gives $g(x_k) = 0 \neq 1/2 = g(1)$, and $x_k = 1/k \to 0$ gives $h(x_k) = 1 \neq 0 = h(0)$. By the contrapositive of the Continuous Limit Theorem (Theorem 6.2.6), neither convergence can be uniform on $[0, \infty)$.
(c) For (g_n) take $[0, a]$ with $0 < a < 1$. For $x \in [0, a]$,

$$|g_n(x) - x| = \frac{x\,|1 - (1 + x^n)|}{1 + x^n} = \frac{x^{n+1}}{1 + x^n} \leq a^{n+1},$$

and $a^{n+1} \to 0$ because $0 < a < 1$ (Example 2.5.3). Given $\epsilon > 0$, any N with $a^{N+1} < \epsilon$ works for all $x \in [0, a]$ simultaneously, so $g_n \to g$ uniformly on $[0, a]$.
For (h_n) take $[a, \infty)$ with $a > 0$. If $n > 1/a$ then $x \geq a > 1/n$ for every x in the set, so $h_n(x) = 1 = h(x)$; that is,

$$\sup_{x \in [a, \infty)} |h_n(x) - h(x)| = 0 \quad \text{for all } n > 1/a,$$

and the convergence is uniform (indeed eventually exact) on $[a, \infty)$.

Problem 6.2.4. Review Exercise 5.2.8, which includes the definition of a uniformly differentiable function: given a differentiable $f : A \to \mathbf{R}$, we say f is uniformly differentiable on A if, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon \quad \text{whenever } 0 < |x - y| < \delta$$

(with $x, y \in A$). Use the results discussed in Section 6.2 to show that if f is uniformly differentiable on an interval A, then f' is continuous on A.

Solution. f' is a uniform limit of continuous difference-quotient functions, so Theorem 6.2.6 gives the conclusion. Continuity is local, so fix $c \in A$ and produce a relative neighborhood of c on which f' is continuous.
Since A is a nondegenerate interval there is a $d \in A$ with $d \neq c$; take $s = +1$ and some $d > c$ unless c is the right endpoint of A, in which case take $s = -1$ and some $d < c$. Put $\eta = |d - c|/2$ and $J = A \cap [c - \eta,\, c + \eta]$, a neighborhood of c relative to A. For $y \in J$ and $0 < t \leq \eta$ the point $y + st$ lies between y and d, hence in A:

$$y + st \in A \quad \text{for all } y \in J \text{ and } 0 < t \leq \eta.$$

For $n > 1/\eta$ set

$$f_n(y) = \frac{f(y + \frac{s}{n}) - f(y)}{s/n} \qquad (y \in J),$$

which is continuous on J: f is continuous (Theorem 5.2.3, f being differentiable), and translation, subtraction and a nonzero scalar are covered by Theorem 4.3.4.
Given $\epsilon > 0$, take $\delta > 0$ from uniform differentiability. For $n > \max\{1/\eta,\, 1/\delta\}$ and $y \in J$, the pair $x = y + \frac{s}{n}$, y has $0 < |x - y| = \frac{1}{n} < \delta$, so $|f_n(y) - f'(y)| < \epsilon$, a bound independent of y. Thus $f_n \to f'$ uniformly on J (Definition 6.2.3), and f' is continuous on J by Theorem 6.2.6, in particular at c.

Problem 6.2.5. Using the Cauchy Criterion for convergent sequences of real numbers (Theorem 2.6.4), supply a proof for Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence): a sequence of functions (f_n) defined on a set $A \subseteq \mathbf{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$. (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Solution. ($\Leftarrow$) The candidate is $f(x) = \lim_{n\to\infty} f_n(x)$: for each fixed $x \in A$ the uniform Cauchy condition says in particular that the real sequence $(f_n(x))$ is Cauchy, hence convergent by Theorem 2.6.4. This defines f on all of A.
Now let $\epsilon > 0$ and choose N so that

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \qquad \text{for all } m, n \geq N \text{ and all } x \in A.$$

Fix $n \geq N$ and $x \in A$. For every $m \geq N$ the displayed bound reads

$$-\tfrac{\epsilon}{2} \;<\; f_n(x) - f_m(x) \;<\; \tfrac{\epsilon}{2},$$

and $f_n(x) - f_m(x) \to f_n(x) - f(x)$ as $m \to \infty$ by the Algebraic Limit Theorem (Theorem 2.3.3). Comparing with the constant sequences $\pm\epsilon/2$, the Order Limit Theorem (Theorem 2.3.4) carries both bounds into the limit as weak inequalities:

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

The single N works for every $x \in A$, so $f_n \to f$ uniformly on A (Definition 6.2.3).
($\Rightarrow$) Assume $f_n \to f$ uniformly on A and let $\epsilon > 0$. Choose N with $|f_n(x) - f(x)| < \epsilon/2$ for all $n \geq N$ and all $x \in A$. Then for $m, n \geq N$ and $x \in A$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Problem 6.2.6. Assume $f_n \to f$ on a set A. Theorem 6.2.6 is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.
(a) If each f_n is uniformly continuous, then f is uniformly continuous.
(b) If each f_n is bounded, then f is bounded.
(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.
(d) If each f_n has fewer than M discontinuities (where $M \in \mathbf{N}$ is fixed), then f has fewer than M discontinuities.
(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Solution. Under uniform convergence (a), (b), (d) and (e) are true and (c) is false. Pointwise counterexamples first.
(a) On $A = [0, 1]$ let $f_n(x) = x^n$. Each f_n is continuous on a compact set, hence uniformly continuous (Theorem 4.4.7). The pointwise limit is $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$, which is not even continuous at 1, let alone uniformly continuous.
(b) On $A = (0, 1]$ let $f_n(x) = \min\{n, 1/x\}$. Each f_n is bounded by n, and for fixed x every $n > 1/x$ gives $f_n(x) = 1/x$, so the pointwise limit $f(x) = 1/x$ is unbounded on $(0, 1]$.
(c) Covered by the counterexample in the uniform case below, which converges uniformly and so a fortiori pointwise.
(d) On $A = \mathbf{R}$ let $f_n(x) = x^{2n}/(1+x^{2n})$ and $M = 2$. Each f_n is continuous, so has $0 < 2$ discontinuities, but

$$f(x) = \begin{cases} 0 & |x| < 1 \\ 1/2 & |x| = 1 \\ 1 & |x| > 1 \end{cases}$$

has exactly 2 discontinuities, at $x = \pm 1$.
(e) On $A = \mathbf{R}$ fix an enumeration $\mathbf{Q} = \{q_1, q_2, q_3, \ldots\}$ (possible by Theorem 1.5.6(i)) and let $f_n =$

$\mathbf{1}_{\{q_1,\dots,q_n\}}$. Each f_n has n discontinuities. The pointwise limit is Dirichlet's function $\mathbf{1}_{\mathbf{Q}}$, which is discontinuous at every real number – an uncountable set of discontinuities.
Now the uniform case. Parts (d) and (e) run on a sharpening of Theorem 6.2.6: if $f_n \to f$ uniformly on A and f_n is continuous at c for infinitely many n, then f is continuous at c – apply Theorem 6.2.6 to the subsequence of those f_n, which still converges uniformly to f. Contrapositively, writing $D(g)$ for the discontinuity set of g: if $c \in D(f)$ then $c \in D(f_n)$ for all but finitely many n, so in particular $D(f) \subseteq \bigcup_n D(f_n)$.
(a) True. Given $\epsilon > 0$, choose N with $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in A$, then choose $\delta > 0$ from the uniform continuity of f_N so that $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \epsilon/3$. For $x, y \in A$ with $|x - y| < \delta$,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ &\quad + |f_N(y) - f(y)| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.\end{aligned}$$

(b) True. Choose N with $|f_N(x) - f(x)| < 1$ for all $x \in A$, and let M_0 bound $|f_N|$. Then $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_0$ for all $x \in A$.
(c) False even for uniform convergence. Take g_n from Exercise 6.2.2(b): $g_n(x) = x$ on $\{1, \frac{1}{2}, \dots, \frac{1}{n}\}$ and 0 elsewhere. Each g_n has exactly n discontinuities, and $\sup_x |g_n(x) - g(x)| = 1/(n+1) \to 0$, so the convergence is uniform; but the limit g is discontinuous at each of the infinitely many points $1/k$.
(d) True. Suppose f had at least M discontinuities $c_1, \dots, c_M$. By the sharpening above, for each i there is an N_i with $n \geq N_i$ implying $c_i \in D(f_n)$. Any $n \geq \max\{N_1, \dots, N_M\}$ then gives an f_n with at least M discontinuities, contradicting the hypothesis. Hence f has fewer than M.
(e) True. By the containment above, $D(f) \subseteq \bigcup_{n=1}^{\infty} D(f_n)$. Each $D(f_n)$ is countable or finite, so listing the elements of the nonempty ones row by row indexes the union by a subset of $\mathbf{N} \times \mathbf{N}$; the union is thus countable or finite by the diagonal argument of Theorem 1.5.8(ii), and so is its subset $D(f)$ (Theorem 1.5.7).

Problem 6.2.7. Let f be uniformly continuous on all of $\mathbf{R}$, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbf{R}$.

Solution. The displacement $|(x + \frac{1}{n}) - x| = \frac{1}{n}$ is the same at every x, which is exactly what uniform continuity is built to exploit.
Let $\epsilon > 0$. By uniform continuity (Definition 4.4.4) there is a $\delta > 0$ such that $|u - v| < \delta$ implies $|f(u) - f(v)| < \epsilon$ for all $u, v \in \mathbf{R}$. Choose $N > 1/\delta$ by the Archimedean Property (Theorem 1.4.2). Then for every $n \geq N$ and every $x \in \mathbf{R}$, the points $u = x + \frac{1}{n}$ and $v = x$ satisfy $|u - v| = \frac{1}{n} \leq \frac{1}{N} < \delta$, so

$$|f_n(x) - f(x)| = \left|f\left(x + \tfrac{1}{n}\right) - f(x)\right| < \epsilon.$$

Since N depends on ϵ alone, $f_n \to f$ uniformly on $\mathbf{R}$ (Definition 6.2.3).
For the counterexample take $f(x) = x^2$, which is continuous on $\mathbf{R}$ but not uniformly continuous there (Theorem 4.4.5 with $x_n = n$, $y_n = n + \frac{1}{n}$, as noted following Example 4.4.6). Here

$$f_n(x) - f(x) = \left(x + \tfrac{1}{n}\right)^2 - x^2 = \frac{2x}{n} + \frac{1}{n^2},$$

which is unbounded in x for each fixed n. Concretely $f_n(n) - f(n) = 2 + 1/n^2 > 2$, so no N can force $|f_n(x) - f(x)| < 1$ for all x.

6.2 Exercises 6.2.8–6.2.14

Problem 6.2.8. Let (g_n) be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $(1/g_n)$ converges uniformly on K to $1/g$.

Solution. The whole exercise is the constant $m = \inf_{x \in K} |g(x)|$, and the point is that $m > 0$: the Continuous Limit Theorem (Theorem 6.2.6) makes g, hence $|g|$, continuous on K, and K is compact, so the Extreme Value Theorem (Theorem 4.4.2) supplies $x_0 \in K$ with $|g(x_0)| = m$; since $g(x_0) \neq 0$, $m > 0$.
Choose N_1 so that $|g_n(x) - g(x)| < m/2$ for all $x \in K$ and $n \geq N_1$. Then for such n and x,

$$|g_n(x)| \geq |g(x)| - |g(x) - g_n(x)| > m - \frac{m}{2} = \frac{m}{2} > 0,$$

so $1/g_n$ is defined on all of K once $n \geq N_1$ (uniform convergence is a statement about a tail, so this suffices), and

$$\begin{aligned}\left|\frac{1}{g_n(x)} - \frac{1}{g(x)}\right| &= \frac{|g(x) - g_n(x)|}{|g_n(x)|\,|g(x)|} \\ &\leq \frac{2}{m^2}\,|g_n(x) - g(x)|.\end{aligned}$$

Given $\epsilon > 0$, pick $N \geq N_1$ with $|g_n(x) - g(x)| < m^2\epsilon/2$ for all $x \in K$ and $n \geq N$. Then $|1/g_n(x) - 1/g(x)| < \epsilon$ for all $x \in K$ and $n \geq N$.

Problem 6.2.9. Assume (f_n) and (g_n) are uniformly convergent sequences of functions.
(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.
(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.
(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbf{N}$, then $(f_n g_n)$ does converge uniformly.

Solution. (a) Say $f_n \to f$ and $g_n \to g$ uniformly on the common domain A. Given $\epsilon > 0$, choose N beyond which both $|f_n(x) - f(x)| < \epsilon/2$ and $|g_n(x) - g(x)| < \epsilon/2$ hold for every $x \in A$. Then for $n \geq N$ and every $x \in A$,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon,$$

so $f_n + g_n \to f + g$ uniformly.
(b) Take $A = \mathbf{R}$, $f_n(x) = x$ and $g_n(x) = 1/n$. The sequence (f_n) is constant, hence uniformly convergent to $f(x) = x$, and $g_n \to 0$ uniformly since $\sup_{\mathbf{R}} |g_n - 0| = 1/n$. But $f_n(x)g_n(x) = x/n$ converges pointwise to 0 while

$$\sup_{x \in \mathbf{R}} \left|\frac{x}{n} - 0\right| = \infty \quad \text{for every } n,$$

so the convergence of $(f_n g_n)$ is not uniform.
(c) The bound $|f_n| \leq M$ passes to the pointwise limit by the Order Limit Theorem (Theorem 2.3.4), so $|f| \leq M$ and likewise $|g| \leq M$. For every $x \in A$,

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)|\,|g_n(x) - g(x)| \\ &\quad + |g(x)|\,|f_n(x) - f(x)| \\ &\leq M\big(|g_n(x) - g(x)| + |f_n(x) - f(x)|\big).\end{aligned}$$

Given $\epsilon > 0$, choose N beyond which $|f_n - f| < \epsilon/(2M)$ and $|g_n - g| < \epsilon/(2M)$ uniformly on A; then $|f_n g_n - fg| < \epsilon$ on A for all $n \geq N$.

Problem 6.2.10. This exercise and the next explore partial converses of the Continuous Limit Theorem (Theorem 6.2.6). Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Solution. Monotonicity lets the two endpoints of a short subinterval control f_n on all of it. First, f is itself increasing: if $x \le y$ then $f_n(x) \le f_n(y)$ for every n, and the inequality survives the pointwise limit (Order Limit Theorem, Theorem 2.3.4).
Since f is continuous on the compact set $[a, b]$, it is uniformly continuous there (Theorem 4.4.7). Let $\epsilon > 0$ and choose $\delta > 0$ so that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta$ in $[a, b]$. Fix a partition

$$a = x_0 < x_1 < \cdots < x_m = b, \qquad x_j - x_{j-1} < \delta,$$

so that $f(x_j) - f(x_{j-1}) < \epsilon/2$ for each j. Because the partition is finite, pointwise convergence gives an N with

$$|f_n(x_j) - f(x_j)| < \frac{\epsilon}{2} \quad \text{for all } n \ge N,\ 0 \le j \le m.$$

Now take any $x \in [a, b]$ and $n \ge N$; say $x \in [x_{j-1}, x_j]$. Since f is increasing, $f(x_{j-1}) \le f(x) \le f(x_j)$, so both $f(x_j) \le f(x) + \epsilon/2$ and $f(x_{j-1}) \ge f(x) - \epsilon/2$. Monotonicity of f_n then gives

$$\begin{aligned} f_n(x) &\le f_n(x_j) < f(x_j) + \frac{\epsilon}{2} \le f(x) + \epsilon, \\ f_n(x) &\ge f_n(x_{j-1}) > f(x_{j-1}) - \frac{\epsilon}{2} \ge f(x) - \epsilon. \end{aligned}$$

Hence $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ and all $n \ge N$.

Problem 6.2.11. (Dini's Theorem). Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.
(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).
(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \ge \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, and use this observation to finish the argument.

Solution. (a) Each $g_n = f - f_n$ is continuous on K (Algebraic Continuity Theorem, Theorem 4.3.4), and for each fixed $x \in K$:

$$g_n(x) \ge 0, \qquad g_{n+1}(x) \le g_n(x), \qquad g_n(x) \to 0.$$

Nonnegativity and monotone decrease both come from $f_n(x) \uparrow f(x)$; the limit is 0 by definition of pointwise convergence. The goal is now: $g_n \to 0$ uniformly on K, i.e. for each $\epsilon > 0$ there is an N with $g_n(x) < \epsilon$ for all $x \in K$ and $n \ge N$.
(b) Fix $\epsilon > 0$. Each K_n is compact: it is bounded, being a subset of the bounded set K; and it is closed, because if $x_k \in K_n$ with $x_k \to x$, then $x \in K$ (K is closed, by Theorem 3.3.4) and $g_n(x) = \lim_k g_n(x_k) \ge \epsilon$ by continuity (Theorem 4.3.2) and the Order Limit Theorem (Theorem 2.3.4), so $x \in K_n$. Theorem 3.3.4 then gives compactness. Nesting is immediate from $g_{n+1} \le g_n$:

$$x \in K_{n+1} \implies \epsilon \le g_{n+1}(x) \le g_n(x) \implies x \in K_n.$$

Suppose, for contradiction, that every K_n is nonempty. The Nested Compact Set Property (Theorem 3.3.5) then produces

$$x_0 \in \bigcap_{n=1}^{\infty} K_n, \qquad \text{so } g_n(x_0) \ge \epsilon \text{ for all } n,$$

contradicting $g_n(x_0) \to 0$. Therefore $K_N = \emptyset$ for some N, and by nesting $K_n = \emptyset$ for all $n \ge N$. That is exactly

$$0 \le g_n(x) < \epsilon \quad \text{for all } x \in K,\ n \ge N,$$

so $g_n \to 0$ uniformly and $f_n \to f$ uniformly on K.

Problem 6.2.12. (Cantor Function). Review the construction of the Cantor set $C \subseteq [0,1]$ from Section 3.1, where $C_0 = [0,1]$, C_n is obtained from C_{n-1} by deleting the open middle third of each of its 2^{n-1} closed intervals, and $C = \bigcap_{n=0}^{\infty} C_n$. This exercise makes use of results and notation from this discussion.
(a) Define $f_0(x) = x$ for all $x \in [0,1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \le x \le 1/3, \\ 1/2 & \text{for } 1/3 < x < 2/3, \\ (3/2)x - 1/2 & \text{for } 2/3 \le x \le 1. \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.
(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \le x \le 1/3, \\ f_1(x) & \text{for } 1/3 < x < 2/3, \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \le x \le 1. \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0,1]$.
(c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Solution. (a) f_0 is the diagonal of the unit square; f_1 is the piecewise linear graph rising with slope $3/2$ from $(0,0)$ to $(1/3, 1/2)$, running flat at height $1/2$ across $(1/3, 2/3)$, then rising with slope $3/2$ from $(2/3, 1/2)$ to $(1,1)$. The three formulas agree at the joints ($(3/2)(1/3) = 1/2$ and $(3/2)(2/3) - 1/2 = 1/2$), so f_1 is continuous; it is increasing because each piece is increasing or constant and the pieces match; and it equals $1/2$ throughout $(1/3, 2/3) = [0,1] \setminus C_1$.
(b) The process is the recursion

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & 0 \le x \le 1/3, \\ 1/2 & 1/3 < x < 2/3, \\ \frac{1}{2} f_n(3x-2) + \frac{1}{2} & 2/3 \le x \le 1, \end{cases}$$

which reproduces the printed f_1 from f_0 and the printed f_2 from f_1 (for $n \ge 1$ the middle clause $f_n(x)$ equals $1/2$ on $(1/3, 2/3)$, so the two forms coincide).
An induction using the recursion gives $f_n(0) = 0$, $f_n(1) = 1$, each f_n continuous and increasing on $[0,1]$: the left piece maps $[0, 1/3]$ increasingly onto $[0, 1/2]$, the right piece maps $[2/3, 1]$ increasingly onto $[1/2, 1]$, and the endpoint values $\frac{1}{2} f_n(1) = \frac{1}{2}$ and $\frac{1}{2} f_n(0) + \frac{1}{2} = \frac{1}{2}$ match the middle constant.
For the convergence, subtract two consecutive instances of the recursion. On $(1/3, 2/3)$ both f_{n+1} and f_n equal $1/2$ (for $n \ge 1$), and on the two outer thirds the difference is $\frac{1}{2}$ times a difference of f_n and f_{n-1} evaluated at $3x$ or $3x - 2$. Hence, writing $\|h\| = \sup_{[0,1]} |h|$,

$$\|f_{n+1} - f_n\| \le \tfrac{1}{2} \|f_n - f_{n-1}\| \quad (n \ge 1).$$

Since $\|f_1 - f_0\| = 1/6$ (Check! the three pieces give $x/2$, $1/2 - x$, $x/2 - 1/2$), induction yields

$$\|f_{n+1} - f_n\| \le \frac{1}{6} \cdot \frac{1}{2^n},$$

and therefore, for $m > n$,

$$\|f_m - f_n\| \le \sum_{k=n}^{m-1} \frac{1}{6 \cdot 2^k} < \frac{1}{3 \cdot 2^n}.$$

Given $\epsilon > 0$, choose N with $3^{-1}2^{-N} < \epsilon$; then $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in [0,1]$ and all $m, n \geq N$. By the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5), (f_n) converges uniformly on $[0,1]$.
(c) Each f_n is continuous and the convergence is uniform, so f is continuous by the Continuous Limit Theorem (Theorem 6.2.6). Each f_n is increasing, so $x \leq y$ gives $f_n(x) \leq f_n(y)$, and the Order Limit Theorem (Theorem 2.3.4) passes this to $f(x) \leq f(y)$. Since $f_n(0) = 0$ and $f_n(1) = 1$ for every n, also $f(0) = 0$ and $f(1) = 1$.
For the derivative, the claim is:
(i) for every $n \geq 1$ and every $m \geq n$, f_m is constant on each connected component of $[0,1] \setminus C_n$.
Induct on n. For $n = 1$ the only component is $(1/3, 2/3)$, where $f_m = 1/2$ for all $m \geq 1$. Assume (i) for n. The components of $[0,1] \setminus C_{n+1}$ are $(1/3, 2/3)$ together with the images of the components of $[0,1] \setminus C_n$ under $x \mapsto x/3$ and $x \mapsto (x+2)/3$. Let $m \geq n+1$. On $(1/3, 2/3)$, $f_m = 1/2$. If I is a component of $[0,1] \setminus C_n$ and $x \in I/3$, then $f_m(x) = \frac{1}{2} f_{m-1}(3x)$ with $3x \in I$ and $m - 1 \geq n$, so f_{m-1} is constant on I by the inductive hypothesis and f_m is constant on $I/3$; the right-hand copy is identical with $f_m(x) = \frac{1}{2} f_{m-1}(3x - 2) + \frac{1}{2}$. This proves (i).
Now let $x \in [0,1] \setminus C$. Then $x \notin C_n$ for some n, so x lies in an open component I of $[0,1] \setminus C_n$. By (i), f_m is constant on I for all $m \geq n$, hence $f = \lim_m f_m$ is constant on I. Since I is an open interval containing x, the difference quotients $(f(y) - f(x))/(y - x)$ vanish for all $y \in I$ with $y \neq x$, so $f'(x) = 0$.

Problem 6.2.13. Recall that the Bolzano-Weierstrass Theorem (Theorem 2.5.5) states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general, but under stronger hypotheses several different conclusions are possible. One avenue is to assume the common domain for all of the functions in the sequence is countable. (Another is explored in the next two exercises.)
Let $A = \{x_1, x_2, x_3, \ldots\}$ be a countable set. For each $n \in \mathbf{N}$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in A$. Follow these steps to show that there exists a subsequence of (f_n) that converges pointwise on A.
(a) Why does the sequence of real numbers $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k})? To indicate that the subsequence of functions (f_{n_k}) is generated by considering the values of the functions at x_1, we will use the notation $f_{n_k} = f_{1,k}$.
(b) Now, explain why the sequence $f_{1,k}(x_2)$ contains a convergent subsequence.
(c) Carefully construct a nested family of subsequences $(f_{m,k})$, and show how this can be used to produce a single subsequence of (f_n) that converges at every point of A.

Solution. (a) The real sequence $\big(f_n(x_1)\big)_n$ satisfies $|f_n(x_1)| \leq M$, so it is bounded, and the Bolzano-Weierstrass Theorem (Theorem 2.5.5) supplies a convergent subsequence $\big(f_{n_k}(x_1)\big)_k$. Relabel the underlying functions $f_{1,k} = f_{n_k}$.
(b) The same bound applies at x_2: $|f_{1,k}(x_2)| \leq M$ for all k, so $\big(f_{1,k}(x_2)\big)_k$ is a bounded real sequence and Theorem 2.5.5 applies again.
(c) Construct rows inductively. Row 1 is $(f_{1,k})_k$ from (a). Given row m, apply Theorem 2.5.5 to the bounded real sequence $\big(f_{m,k}(x_{m+1})\big)_k$ and let row $m+1$, $(f_{m+1,k})_k$, be the corresponding subsequence of $(f_{m,k})_k$. Two properties hold by construction:

$$\begin{aligned}&\text{(i) } (f_{m+1,k})_k \text{ is a subsequence of } (f_{m,k})_k,\\ &\text{(ii) } \big(f_{m,k}(x_j)\big)_k \text{ converges for every } j \leq m.\end{aligned}$$

Property (ii) follows from (i) together with Theorem 2.5.2: row m was chosen to converge at x_m, and every later row is a subsequence of it, so convergence at $x_1, \ldots, x_{m-1}$ inherited from earlier rows is not lost.
Now take the diagonal: $g_k = f_{k,k}$. It is a genuine subsequence of (f_n): row $k+1$ is a subsequence of row k, so its $(k+1)$-st term sits at position $\geq k+1$ in row k, whence $f_{k+1,k+1}$ carries a strictly larger original index than $f_{k,k}$.
Fix $m \in \mathbf{N}$. For $k \geq m$, the function $g_k = f_{k,k}$ is a term of row k, which is a subsequence of row m; moreover the terms $g_m, g_{m+1}, g_{m+2}, \ldots$ occur in row m in strictly increasing order of position. Hence

$(g_k)_{k\geq m}$ is a subsequence of $(f_{m,k})_k$, and by (ii) and Theorem 2.5.2 the sequence $\big(g_k(x_m)\big)_k$ converges. Since m was arbitrary, (g_k) converges at every point of A.

Problem 6.2.14. A sequence of functions (f_n) defined on a set $E \subseteq \mathbf{R}$ is called *equicontinuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbf{N}$ and $|x-y| < \delta$ in E.
(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?
(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0,1]$. Is each g_n uniformly continuous on $[0,1]$?

Solution. (a) The position of the quantifier on n. Uniform continuity of each f_n separately says

$$\begin{aligned}&\forall \epsilon > 0 \;\; \forall n \;\; \exists \delta_n > 0 :\\ &\qquad |x-y| < \delta_n \Rightarrow |f_n(x) - f_n(y)| < \epsilon,\end{aligned}$$

so δ is allowed to depend on n. Equicontinuity says

$$\begin{aligned}&\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall n :\\ &\qquad |x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon,\end{aligned}$$

one δ serving the whole family at once. Equicontinuity therefore implies each f_n is uniformly continuous, and the converse fails as soon as the admissible δ_n can only be chosen with $\inf_n \delta_n = 0$.
(b) Yes, each g_n is uniformly continuous, being continuous on the compact set $[0,1]$ (Theorem 4.4.7). But the family is not equicontinuous: near $x = 1$ the graphs steepen without bound ($g_n'(1) = n$), so the δ that works for g_n shrinks to 0 as n grows. Explicitly, with $\epsilon_0 = 1/4$,

$$y_n = 2^{-1/n}, \qquad g_n(y_n) = \tfrac{1}{2}, \qquad g_n(1) = 1,$$

and $y_n \to 1$, so every proposed $\delta > 0$ admits an n with $|1 - y_n| < \delta$ while $|g_n(1) - g_n(y_n)| = 1/2 > \epsilon_0$.

6.3 Exercises 6.2.15–6.3.6

Problem 6.2.15. (Arzela-Ascoli Theorem). For each $n \in \mathbf{N}$, let f_n be a function defined on $[0,1]$. If (f_n) is bounded on $[0,1]$ — that is, there exists an $M > 0$ such that $|f_n(x)| \le M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$ — and if the collection of functions (f_n) is equicontinuous (Exercise 6.2.14), follow these steps to show that (f_n) contains a uniformly convergent subsequence.
(a) Use Exercise 6.2.13 to produce a subsequence (f_{n_k}) that converges at every rational point in $[0,1]$. To simplify the notation, set $g_k = f_{n_k}$. It remains to show that (g_k) converges uniformly on all of $[0,1]$.
(b) Let $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}$$

for all $|x-y| < \delta$ and $k \in \mathbf{N}$. Using this δ, let $r_1, r_2, \ldots, r_m$ be a *finite* collection of rational points with the property that the union of the neighborhoods $V_\delta(r_i)$ contains $[0,1]$.
Explain why there must exist an $N \in \mathbf{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3}$$

for all $s, t \ge N$ and r_i in the finite subset of $[0,1]$ just described. Why does having the set $\{r_1, r_2, \ldots, r_m\}$ be finite matter?
(c) Finish the argument by showing that, for an arbitrary $x \in [0,1]$,

$$|g_s(x) - g_t(x)| < \epsilon$$

for all $s, t \ge N$.

Solution. (a) Apply Exercise 6.2.13 with $A = \mathbf{Q} \cap [0,1]$, enumerated as $A = \{q_1, q_2, q_3, \ldots\}$: this A is countable, being an infinite subset of the countable set $\mathbf{Q}$ (Theorem 1.5.6(i) and Theorem 1.5.7), and the hypothesis $|f_n(x)| \le M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$ restricts to the uniform bound on A that 6.2.13 requires. It therefore supplies a subsequence (f_{n_k}) converging at every point of A. Set $g_k = f_{n_k}$; equicontinuity passes to the subsequence unchanged, since each g_k is one of the f_n and the δ from the definition is common to the whole family.
(b) Fix $\epsilon > 0$ and let $\delta > 0$ be the equicontinuity constant for $\epsilon/3$, so that

$$|x-y| < \delta \implies |g_k(x) - g_k(y)| < \frac{\epsilon}{3} \quad \text{for all } k \in \mathbf{N}.$$

The collection $\{V_\delta(r) : r \in \mathbf{Q} \cap [0,1]\}$ is an open cover of $[0,1]$: for $x \in [0,1]$ the interval $(x-\delta, x+\delta) \cap (0,1)$ is nonempty and open, so density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) puts a rational r inside it, and $|x-r| < \delta$ gives $x \in V_\delta(r)$. As $[0,1]$ is closed and bounded, Heine-Borel (Theorem 3.3.8, (ii) $\Rightarrow$ (iii)) extracts a finite subcover

$$[0,1] \subseteq V_\delta(r_1) \cup V_\delta(r_2) \cup \cdots \cup V_\delta(r_m), \qquad r_i \in \mathbf{Q} \cap [0,1].$$

Each r_i is rational, so by part (a) the real sequence $(g_k(r_i))_{k=1}^\infty$ converges, hence is Cauchy (Theorem 2.6.2): for each $i \in \{1, \ldots, m\}$ there is an $N_i \in \mathbf{N}$ with $|g_s(r_i) - g_t(r_i)| < \epsilon/3$ for all $s, t \ge N_i$. Put

$$N = \max\{N_1, N_2, \ldots, N_m\},$$

which then works simultaneously for all m points.
Finiteness is what makes this last step legal: a maximum of finitely many natural numbers is again a natural number, whereas the infinite list $N_1, N_2, N_3, \ldots$ attached to all of $\mathbf{Q} \cap [0,1]$ need not be bounded above, and then no single N would serve every rational at once.
(c) Let $x \in [0,1]$ be arbitrary and choose i with $x \in V_\delta(r_i)$, i.e. $|x - r_i| < \delta$. For $s, t \ge N$,

$$\begin{aligned} |g_s(x) - g_t(x)| &\le |g_s(x) - g_s(r_i)| \\ &\quad + |g_s(r_i) - g_t(r_i)| \\ &\quad + |g_t(r_i) - g_t(x)| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon, \end{aligned}$$

the outer two terms by the choice of δ in (b) (applied to the functions g_s and g_t) and the middle term by the choice of N. The index N depends only on ϵ, not on x, so the estimate holds for every $x \in [0,1]$ at once. By the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5), $(g_k) = (f_{n_k})$ converges uniformly on $[0,1]$.

Problem 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.
(b) Now, show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

Solution. (a) $g \equiv 0$ and $g' \equiv 0$; the convergence is uniform because

$$\sup_{x\in[0,1]} \left| \frac{x^n}{n} - 0 \right| = \frac{1}{n} \longrightarrow 0.$$

The zero function is differentiable with $g'(x) = 0$ for every $x \in [0,1]$ (one-sided at the endpoints).
(b) $g_n'(x) = x^{n-1}$, which converges pointwise on $[0,1]$ to

$$h(x) = \begin{cases} 0 & 0 \le x < 1, \\ 1 & x = 1. \end{cases}$$

The convergence is **not** uniform: each g_n' is continuous on $[0,1]$ while h is not, so the Continuous Limit Theorem (Theorem 6.2.6) forbids uniform convergence.
So h and g' agree on $[0,1)$ but not at the endpoint: $h(1) = 1 \neq 0 = g'(1)$. They are not the same.

Problem 6.3.2. Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \tfrac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbf{R}$.
(b) Note that each h_n is differentiable. Show that $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is **not** uniform on any neighborhood of zero.

Solution. (a) $h_n \to h$ uniformly on $\mathbf{R}$, where $h(x) = |x|$. Rationalizing the difference,

$$\begin{aligned} \left| \sqrt{x^2 + \tfrac{1}{n}} - |x| \right| &= \frac{1/n}{\sqrt{x^2 + \tfrac{1}{n}} + |x|} \\ &\le \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}}, \end{aligned}$$

uniformly in x, and $1/\sqrt{n} \to 0$.
(b) Since $x^2 + 1/n > 0$ for all x, each h_n is differentiable with

$$h_n'(x) = \frac{x}{\sqrt{x^2 + \tfrac{1}{n}}}.$$

For $x \neq 0$ the denominator tends to $|x|$, so $h_n'(x) \to x/|x|$; and $h_n'(0) = 0$ for every n. Hence

$$g(x) = \lim h_n'(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

Each h_n' is continuous, but g is discontinuous at 0. By the Continuous Limit Theorem (Theorem 6.2.6), uniform convergence on a set forces the limit to be continuous there; so the convergence cannot be uniform on any neighborhood of 0.

Problem 6.3.3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

(a) Find the points on $\mathbf{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbf{R}$. What is the limit function?
(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution. (a) The extrema sit at $x = \pm 1/\sqrt{n}$, and $f_n \to 0$ uniformly. Indeed

$$f_n'(x) = \frac{(1+nx^2) - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2},$$

which vanishes exactly at $x = \pm 1/\sqrt{n}$, is positive on $(-1/\sqrt{n}, 1/\sqrt{n})$ and negative outside. So f_n has its minimum at $x = -1/\sqrt{n}$ and its maximum at $x = 1/\sqrt{n}$, with

$$f_n\Big(\tfrac{1}{\sqrt{n}}\Big) = \frac{1/\sqrt{n}}{2} = \frac{1}{2\sqrt{n}}, \qquad f_n\Big(\tfrac{-1}{\sqrt{n}}\Big) = \frac{-1}{2\sqrt{n}}.$$

(Also $f_n(x) \to 0$ as $x \to \pm\infty$, so these are global extrema.) Hence

$$\sup_{x \in \mathbf{R}} |f_n(x) - 0| = \frac{1}{2\sqrt{n}} \longrightarrow 0,$$

so $f_n \to f \equiv 0$ uniformly on $\mathbf{R}$.
(b) With f_n' as computed above, fix $x \neq 0$: the numerator grows like $-nx^2$ while the denominator grows like n^2x^4, so

$$\lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} = 0 = f'(x).$$

At $x = 0$, however, $f_n'(0) = 1$ for every n, so $\lim f_n'(0) = 1 \neq 0 = f'(0)$.
Thus $f'(x) = \lim f_n'(x)$ for all $x \neq 0$, and only there.

Problem 6.3.4. Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbf{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbf{R}$.

Solution. Uniform convergence is the bound $|\sin(nx)| \le 1$:

$$\sup_{x\in\mathbf{R}} \left| \frac{\sin(nx)}{\sqrt{n}} - 0 \right| \le \frac{1}{\sqrt{n}} \longrightarrow 0.$$

For the derivatives, $h_n'(x) = \sqrt{n}\cos(nx)$. Fix $x \in \mathbf{R}$ and suppose, for contradiction, that $(h_n'(x))$ converges. A convergent sequence is bounded (Theorem 2.3.2), so there is $M > 0$ with $\sqrt{n}\,|\cos(nx)| \le$

M for all n, whence

$$|\cos(nx)| \le \frac{M}{\sqrt{n}} \longrightarrow 0, \qquad \text{so} \qquad \cos(nx) \to 0.$$

Now read off the same limit two ways along the even indices. On the one hand $(\cos(2nx))$ is a subsequence of $(\cos(nx))$, so $\cos(2nx) \to 0$ (Theorem 2.5.2). On the other hand the double-angle identity gives

$$\cos(2nx) = 2\cos^2(nx) - 1 \longrightarrow 2 \cdot 0 - 1 = -1.$$

Uniqueness of limits (Theorem 2.2.7) makes $0 = -1$, a contradiction. Hence $(h_n'(x))$ diverges, and x was arbitrary.

Problem 6.3.5. Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:
(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.
(b) Compute $g_n'(x)$ for each $n \in \mathbf{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.
(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution. (a) $g(x) = x/2$, so $g'(x) = 1/2$. Indeed $g_n(x) = \dfrac{x}{2} + \dfrac{x^2}{2n} \to \dfrac{x}{2}$ for each fixed x.
(b) $g_n'(x) = \dfrac{1}{2} + \dfrac{x}{n}$, and on $[-M, M]$

$$\sup_{|x| \le M} \left|g_n'(x) - \tfrac{1}{2}\right| = \frac{M}{n} \longrightarrow 0,$$

so $g_n' \to 1/2$ uniformly there. Each g_n is a polynomial, hence differentiable on $[-M, M]$, and $g_n(0) = 0$ converges, so Theorem 6.3.3 applies with $x_0 = 0$: $g = \lim g_n$ is differentiable on $[-M, M]$ with $g'(x) = \lim g_n'(x) = 1/2$. Since M was arbitrary and differentiability is local, this holds on all of $\mathbf{R}$.
(c) $f(x) = x^2/2$ and $f'(x) = x$. For (a), fix x and divide through by n:

$$f_n(x) = \frac{x^2 + 1/n}{2 + x/n} \longrightarrow \frac{x^2}{2}.$$

For (b), fix $M > 0$ and restrict to $[-M, M]$ with $n > M$, so that $2n + x \ge 2n - M > n > 0$ and f_n is a well-defined differentiable rational function there. The quotient rule gives

$$\begin{aligned} f_n'(x) &= \frac{2nx(2n + x) - (nx^2 + 1)}{(2n + x)^2} \\ &= \frac{4n^2x + nx^2 - 1}{(2n + x)^2}, \end{aligned}$$

and subtracting x (using $x(2n + x)^2 = 4n^2x + 4nx^2 + x^3$),

$$f_n'(x) - x = \frac{-\left(3nx^2 + x^3 + 1\right)}{(2n + x)^2}.$$

Hence for $|x| \le M$ and $n > M$,

$$\left|f_n'(x) - x\right| \le \frac{3nM^2 + M^3 + 1}{n^2} \longrightarrow 0,$$

so (f_n') converges uniformly on $[-M, M]$ to the identity function. With $f_n(0) = 1/(2n) \to 0$ convergent, Theorem 6.3.3 gives that $f = \lim f_n$ is differentiable on $[-M, M]$ with $f'(x) = \lim f_n'(x) = x$, matching (a).

Problem 6.3.6. Provide an example or explain why the request is impossible. Let us take the domain of the functions to be all of $\mathbf{R}$.
(a) A sequence (f_n) of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.
(b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbf{R}$.
(c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Solution. (a) Possible: take $f_n = \dfrac{1}{n} g$, where g is the continuous nowhere-differentiable function of Section 5.4,

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} h(2^k x),$$

with h the 2-periodic extension of $|x|$ on $[-1, 1]$. Each f_n is nowhere differentiable (if f_n were differentiable at c, so would be $nf_n = g$). Since $0 \le h \le 1$ we have $|g| \le \sum_k 2^{-k} = 2$, so

$$\sup_{x \in \mathbf{R}} |f_n(x) - 0| \le \frac{2}{n} \longrightarrow 0,$$

giving $f_n \to f \equiv 0$ uniformly, and f is everywhere differentiable.
(b) Possible: take the constant functions $f_n(x) = n$. Then $f_n' \equiv 0$, so (f_n') converges uniformly to 0, while $(f_n(x)) = (n)$ diverges for every $x \in \mathbf{R}$.
(c) Impossible. If (f_n) converges uniformly then in particular $f_n \to f$ pointwise, and (f_n') converges uniformly to some g. Fix any $c \in \mathbf{R}$ and apply the Differentiable Limit Theorem (Theorem 6.3.1) on the closed interval $[c-1, c+1]$: its hypotheses hold there (pointwise convergence of (f_n), differentiability of each f_n, uniform convergence of (f_n') to g), so f is differentiable on $[c-1, c+1]$ with $f' = g$. In particular f is differentiable at c, and c was arbitrary.

6.4 Exercises 6.3.7–6.4.6

Problem 6.3.7. Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n \in \mathbf{N}$,

$$\begin{aligned} |f_n(x) - f_m(x)| \le {} & |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| \\ & + |f_n(x_0) - f_m(x_0)|. \end{aligned}$$

(Theorem 6.3.2: Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f_n') converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.)

Solution. We show (f_n) is uniformly Cauchy on $[a, b]$; the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5) then gives the conclusion. Assume $a < b$, the case $a = b$ being trivial. Let $\epsilon > 0$.
(i) $(f_n(x_0))$ converges, hence is Cauchy, so there is N_1 with

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad \text{for all } m, n \ge N_1.$$

(ii) (f_n') converges uniformly on $[a, b]$, hence is uniformly Cauchy (Theorem 6.2.5 again), so there is N_2 with

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)} \qquad \text{for all } t \in [a, b],\ m, n \ge N_2.$$

Set $N = \max\{N_1, N_2\}$ and fix $m, n \ge N$ and $x \in [a, b]$. Put $d = f_n - f_m$, differentiable and hence continuous on $[a, b]$. If $x \ne x_0$, the Mean Value Theorem (Theorem 5.3.2), applied to d on the closed interval with endpoints x_0 and x, supplies α strictly between them with

$$d(x) - d(x_0) = d'(\alpha)(x - x_0) = \big(f_n'(\alpha) - f_m'(\alpha)\big)(x - x_0).$$

Since $\alpha \in [a, b]$ and $|x - x_0| \le b - a$, step (ii) gives

$$\big|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))\big| < \frac{\epsilon}{2(b-a)}(b - a) = \frac{\epsilon}{2},$$

and this bound holds trivially when $x = x_0$, where the left side is 0.
Feeding this and (i) into the displayed triangle inequality,

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in [a, b]$ whenever $m, n \ge N$.

Problem 6.4.1. Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5): for each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$, and let $M_n > 0$ be a real number satisfying

$$|f_n(x)| \le M_n \quad \text{for all } x \in A.$$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Solution. Verify the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.4.4) directly from the Cauchy Criterion for the convergent numerical series $\sum M_n$.
Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, its sequence of partial sums is Cauchy, so by the Cauchy Criterion for Series (Theorem 2.7.2) there is $N \in \mathbf{N}$ with

$$M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon \qquad \text{whenever } n > m \ge N.$$

(The terms M_k are positive, so the absolute value bars in 2.7.2 may be dropped.) For this same N, any $n > m \ge N$, and any $x \in A$, the triangle inequality gives

$$\begin{aligned} |f_{m+1}(x) + \cdots + f_n(x)| &\le |f_{m+1}(x)| + \cdots + |f_n(x)| \\ &\le M_{m+1} + \cdots + M_n \\ &< \epsilon. \end{aligned}$$

The bound is independent of x, so the hypothesis of Theorem 6.4.4 holds, and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A. □

Problem 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.
(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.
(b) If $0 \le f_n(x) \le g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.
(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \le M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution. (a) True, (b) true, (c) false.
(a) Run the Cauchy Criterion (Theorem 6.4.4) at $n = m + 1$, where the sum collapses to a single term: given $\epsilon > 0$, the N supplied by 6.4.4 satisfies

$$|g_{m+1}(x)| < \epsilon \qquad \text{for all } m \ge N \text{ and all } x \in A,$$

which is uniform convergence of (g_n) to 0.
(b) Again Theorem 6.4.4, with the bars free of charge since $0 \le f_k \le g_k$. Given $\epsilon > 0$, pick N so that

$|g_{m+1}(x) + \cdots + g_n(x)| < \epsilon$ for all $n > m \geq N$ and all x. Then for those same m, n, x,

$$\left|\sum_{k=m+1}^{n} f_k(x)\right| = \sum_{k=m+1}^{n} f_k(x) \leq \sum_{k=m+1}^{n} g_k(x)$$
$$= \left|\sum_{k=m+1}^{n} g_k(x)\right| < \epsilon.$$

So $\sum f_n$ meets the Cauchy Criterion and converges uniformly.
(c) False. On $A = [0,1]$ take the constant functions $f_n(x) = (-1)^n/n$. The numerical series $\sum(-1)^n/n$ converges by the Alternating Series Test (Theorem 2.7.7: the terms $1/n$ decrease to 0), and since every partial sum s_k is constant, $\sup_{x\in A}|s_k(x) - f(x)|$ is just the numerical error $|s_k - f| \to 0$, so the convergence is uniform. But any admissible bound obeys

$$M_n \geq \sup_{x\in A}|f_n(x)| = \frac{1}{n},$$

and $\sum 1/n$ diverges (Example 2.4.5), so $\sum M_n$ diverges by comparison (Theorem 2.7.4).

Problem 6.4.3. (a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbf{R}$.
(b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Solution. (a) The M-Test with $M_n = 1/2^n$. For every $x \in \mathbf{R}$,

$$\left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n} = M_n, \qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 < \infty$$

(geometric series, Example 2.7.5), so by Corollary 6.4.5 the series converges uniformly on $\mathbf{R}$. Each term $x \mapsto \cos(2^n x)/2^n$ is continuous, so the Term-by-term Continuity Theorem (Theorem 6.4.2) makes g continuous on all of $\mathbf{R}$.
(b) The attempt fails at the one hypothesis that matters: the differentiated series does not converge uniformly on any nondegenerate interval. Writing $f_n(x) = \cos(2^n x)/2^n$, the Chain Rule (Theorem 5.2.5) gives

$$f_n'(x) = \frac{-2^n \sin(2^n x)}{2^n} = -\sin(2^n x),$$

so Theorem 6.4.3 asks for uniform convergence of $\sum_{n=0}^{\infty} -\sin(2^n x)$. Fix any interval $[a,b]$ with $a < b$ and choose N with $2^N(b-a) \geq 2\pi$. For every $n \geq N$ the image of $[a,b]$ under $x \mapsto 2^n x$ is an interval of length at least 2π, hence contains a full period of the sine, and therefore

$$\sup_{x\in[a,b]} |f_n'(x)| = 1 \quad \text{for all } n \geq N.$$

Thus (f_n') does not converge uniformly to 0 on $[a,b]$, so by Exercise 6.4.2(a) the series $\sum f_n'$ cannot converge uniformly on $[a,b]$. Its sole hypothesis unmet, Theorem 6.4.3 yields no information about the differentiability of g.

Problem 6.4.4. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1+x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Solution. The series converges exactly on $(-1,1)$, and g is continuous there.
Write $t = x^2 \geq 0$, so the nth term is $a_n = t^n/(1+t^n)$.
(i) $|x| \geq 1$. Then $t \geq 1$, so $t^n \geq 1$ and

$$a_n = \frac{t^n}{1+t^n} = \frac{1}{1+t^{-n}} \geq \frac{1}{2},$$

so $a_n \not\to 0$ and the series diverges by Theorem 2.7.3.
(ii) $|x| < 1$. Then $1+t^n \geq 1$ gives $0 \leq a_n \leq t^n = x^{2n}$, and $\sum_{n=0}^{\infty} x^{2n}$ is a convergent geometric series ($0 \leq x^2 < 1$), so the series converges by the Comparison Test (Theorem 2.7.4).
For continuity, fix any b with $0 < b < 1$. For all $x \in [-b,b]$ and all $n \geq 0$,

$$\left|\frac{x^{2n}}{1+x^{2n}}\right| \leq x^{2n} \leq b^{2n} =: M_n, \qquad \sum_{n=0}^{\infty} b^{2n} = \frac{1}{1-b^2} < \infty,$$

so by the Weierstrass M-Test (Corollary 6.4.5) the series converges uniformly on $[-b,b]$. Each term is continuous (the denominator $1+x^{2n} \geq 1$ never vanishes), so Theorem 6.4.2 makes g continuous on $[-b,b]$. Given $c \in (-1,1)$, choosing b with $|c| < b < 1$ places c in the interior of $[-b,b]$, so continuity there is continuity of g at c; hence g is continuous on $(-1,1)$.

Problem 6.4.5. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1,1]$.
(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1,1)$ but does not converge when $x=1$. For a fixed $x_0 \in (-1,1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Solution. (a) The M-Test with $M_n = 1/n^2$. For $|x| \leq 1$,

$$\left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2} \leq \frac{1}{n^2} = M_n,$$

and $\sum_{n=1}^{\infty} 1/n^2$ converges (p-series with $p = 2 > 1$, Corollary 2.4.7). By Corollary 6.4.5 the series converges uniformly on $[-1,1]$; each term x^n/n^2 is a polynomial, hence continuous; so h is continuous on $[-1,1]$ by Theorem 6.4.2.
(b) Localize: run the M-Test not on $[-1,1)$ but on $[-b,b]$ with $|x_0| < b < 1$, say $b = (1+|x_0|)/2$. (On all of $[-1,1)$ it must fail, since $\sup_{|x|<1} |x^n/n| = 1/n$ and $\sum 1/n$ diverges.) For every $x \in [-b,b]$ and every $n \in \mathbf{N}$,

$$\left|\frac{x^n}{n}\right| \leq \frac{b^n}{n} \leq b^n =: M_n, \qquad \sum_{n=1}^{\infty} b^n = \frac{b}{1-b} < \infty$$

since $0 < b < 1$. By Corollary 6.4.5 the series $\sum x^n/n$ converges uniformly on $[-b, b]$, and by Theorem 6.4.2 its sum f is continuous on $[-b, b]$.
Since $|x_0| < b$, some neighborhood $V_\delta(x_0)$ lies inside $[-b, b]$, so the $\delta' \leq \delta$ furnished by continuity on $[-b, b]$ serves verbatim for f on $[-1, 1)$: f is continuous at x_0.

Problem 6.4.6. Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} - \cdots .$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Solution. f is defined on all of $(0, \infty)$, and it is continuous and differentiable there, with

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}.$$

The M-Test is unavailable ($\sum 1/(x+n)$ diverges); the alternating-series tail estimate supplies the uniform control instead.
Put $f_n(x) = (-1)^n/(x+n)$ and $s_k = f_0 + \cdots + f_k$. For each fixed $x > 0$ the numbers $b_n = 1/(x+n)$ are positive, decrease in n, and tend to 0, so $\sum_{n=0}^{\infty}(-1)^n b_n$ converges by the Alternating Series Test (Theorem 2.7.7): $f(x)$ exists for every $x > 0$. The nested-interval proof of 2.7.7 (Exercise 2.7.1(b)) traps the limit between consecutive partial sums, whence

$$|f(x) - s_k(x)| \leq |s_{k+1}(x) - s_k(x)| = \frac{1}{x+k+1} \leq \frac{1}{k+1}$$

for all $x > 0$. The bound is independent of x and tends to 0, so $s_k \to f$ uniformly on $(0, \infty)$, and f is continuous there by Theorem 6.4.2 (each s_k is a finite sum of continuous functions).
Each f_n is differentiable on $(0, \infty)$ with

$$f_n'(x) = \frac{(-1)^{n+1}}{(x+n)^2},$$

and $\sum_{n=0}^{\infty} f_n'$ is again alternating with terms $1/(x+n)^2$ positive, decreasing, and tending to 0; the same estimate bounds its tails by $1/(k+1)^2$ uniformly in $x > 0$, so it converges uniformly on $(0, \infty)$ to some g.
Fix any $[a, b] \subset (0, \infty)$. There the f_n are differentiable, $\sum f_n'$ converges uniformly, and $\sum f_n(a)$ converges – every hypothesis of the Term-by-term Differentiability Theorem (Theorem 6.4.3). So f is differentiable on $[a, b]$ with $f' = g$, and since each point of $(0, \infty)$ is interior to such an interval, f is differentiable on all of $(0, \infty)$. □

6.5 Exercises 6.4.7–6.5.3

Problem 6.4.7. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.
(b) Can we determine if f is twice-differentiable?

Solution. (a) $f'(x) = \sum_{k=1}^{\infty} \cos(kx)/k^2$, continuous on $\mathbf{R}$.
Put $f_k(x) = \sin(kx)/k^3$, so $f_k'(x) = k\cos(kx)/k^3 = \cos(kx)/k^2$. Two applications of the M-Test

(Corollary 6.4.5), both with convergent p-series (Corollary 2.4.7):

$$\left|\frac{\sin(kx)}{k^3}\right| \leq \frac{1}{k^3}, \qquad \left|\frac{\cos(kx)}{k^2}\right| \leq \frac{1}{k^2} \qquad (x \in \mathbf{R}),$$

so both $\sum f_k$ and $\sum f_k'$ converge uniformly on $\mathbf{R}$.
On any $[a,b]$ the f_k are differentiable, $\sum f_k'$ converges uniformly, and $\sum f_k(a)$ converges – all hypotheses of Theorem 6.4.3 – so f is differentiable there with $f' = \sum f_k'$; as $[a,b]$ was arbitrary this holds on $\mathbf{R}$. Finally f' is continuous by Theorem 6.4.2, being a uniformly convergent sum of the continuous functions $\cos(kx)/k^2$.
(b) No – not with Theorem 6.4.3, because the next differentiated series genuinely fails to converge uniformly near $x = 0$.
Differentiating once more gives $f_k''(x) = -\sin(kx)/k$, and the M-Test is useless: $\sup_x |\sin(kx)/k| = 1/k$ and $\sum 1/k$ diverges. Uniform convergence genuinely fails on any set $A \supseteq (0, \delta)$, hence on every interval with 0 in its interior – precisely what 6.4.3 would need to reach $x = 0$. Suppose $\sum_{k=1}^{\infty} \sin(kx)/k$ did converge uniformly on such an A. By the Cauchy Criterion (Theorem 6.4.4) with $\epsilon = \sqrt{2}/4$ there would be N with

$$\left|\sum_{k=m+1}^{n} \frac{\sin(kx)}{k}\right| < \frac{\sqrt{2}}{4} \qquad \text{for all } n > m \geq N,\ x \in A.$$

Take $m \geq N$ large enough that $x_m = \pi/(4m) < \delta$ lies in A, and set $n = 2m$. For $m+1 \leq k \leq 2m$ we have $kx_m \in [\pi/4, \pi/2]$, where $\sin \geq \sqrt{2}/2$, so

$$\begin{aligned} \sum_{k=m+1}^{2m} \frac{\sin(kx_m)}{k} &\geq \frac{\sqrt{2}}{2} \sum_{k=m+1}^{2m} \frac{1}{k} \\ &\geq \frac{\sqrt{2}}{2} \cdot m \cdot \frac{1}{2m} = \frac{\sqrt{2}}{4}, \end{aligned}$$

contradicting the displayed bound. So the hypothesis of Theorem 6.4.3 fails, and it tells us nothing about whether f' is differentiable at 0. □

Problem 6.4.8. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution. f is defined on all of $\mathbf{R}$, where it is continuous, differentiable, and twice-differentiable.
Write $g_k(x) = \sin(x/k)/k$. Since $|\sin t| \leq |t|$, for $|x| \leq M$

$$|g_k(x)| = \left|\frac{\sin(x/k)}{k}\right| \leq \frac{|x|}{k^2} \leq \frac{M}{k^2},$$

and $\sum M/k^2$ converges, so the Weierstrass M-Test (Corollary 6.4.5) gives uniform convergence of $\sum g_k$ on $[-M, M]$. Each g_k is continuous, so the Term-by-term Continuity Theorem (Theorem 6.4.2) makes f continuous on $[-M, M]$; since $M > 0$ is arbitrary and continuity is local, f is defined and continuous on all of $\mathbf{R}$.
For the derivative, $g_k'(x) = \cos(x/k)/k^2$ satisfies

$$|g_k'(x)| \leq \frac{1}{k^2} \quad \text{for all } x \in \mathbf{R},$$

so $\sum g_k'$ converges uniformly on $\mathbf{R}$ by the M-Test. Each g_k is differentiable and $\sum g_k(0) = 0$ converges, so the Term-by-term Differentiability Theorem (Theorem 6.4.3), applied on $A = \mathbf{R}$, gives that f is differentiable with

$$f'(x) = \sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}.$$

Run the same argument on this series. Here $g_k''(x) = -\sin(x/k)/k^3$ with

$$|g_k''(x)| \leq \frac{1}{k^3} \quad \text{for all } x \in \mathbf{R},$$

so $\sum g_k''$ converges uniformly on $\mathbf{R}$, while $\sum g_k'(0) = \sum 1/k^2$ converges. Theorem 6.4.3 applies again: f' is differentiable on $\mathbf{R}$, so f is twice-differentiable there, with

$$f''(x) = -\sum_{k=1}^{\infty} \frac{\sin(x/k)}{k^3}.$$

Problem 6.4.9. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbf{R}$.
(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution. (a) Each term $h_n(x) = 1/(x^2 + n^2)$ is defined and continuous on all of $\mathbf{R}$ (the denominator is never zero), and

$$0 < \frac{1}{x^2 + n^2} \leq \frac{1}{n^2} \quad \text{for all } x \in \mathbf{R}.$$

Since $\sum 1/n^2$ converges, the Weierstrass M-Test (Corollary 6.4.5) gives that $\sum h_n$ converges uniformly on $\mathbf{R}$; in particular h is defined on all of $\mathbf{R}$, and the Term-by-term Continuity Theorem (Theorem 6.4.2) makes h continuous there.
(b) Yes, and h' is continuous. We have

$$h_n'(x) = \frac{-2x}{(x^2 + n^2)^2},$$

and this is bounded uniformly in x by a summable constant: for $x \neq 0$, the inequality $x^2 + n^2 \geq 2|x|n$ gives

$$\begin{aligned} |h_n'(x)| &= \frac{2|x|}{(x^2 + n^2)^2} \\ &\leq \frac{2|x|}{\big(2|x|n\big)(x^2 + n^2)} \\ &= \frac{1}{n(x^2 + n^2)} \leq \frac{1}{n^3}, \end{aligned}$$

and $|h_n'(0)| = 0 \leq 1/n^3$ as well. Since $\sum 1/n^3$ converges, the M-Test makes $\sum h_n'$ converge uniformly on $\mathbf{R}$. Each h_n is differentiable and $\sum h_n(0) = \sum 1/n^2$ converges, so Theorem 6.4.3 gives that h is differentiable on $\mathbf{R}$ with

$$h'(x) = -\sum_{n=1}^{\infty} \frac{2x}{(x^2 + n^2)^2}.$$

This last series converges uniformly on $\mathbf{R}$ and its terms are continuous, so h' is continuous by Theorem 6.4.2.

Problem 6.4.10. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the set of rational numbers. For each

$r_n \in \mathbf{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n, \\ 0 & \text{for } x \le r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbf{R}$ that is continuous at every irrational point.

Solution. Everything follows from the single bound $0 \le u_n(x) \le 1/2^n$, valid for every $x \in \mathbf{R}$.
Defined on all of $\mathbf{R}$: since $\sum_{n=1}^{\infty} 1/2^n = 1$ converges, the Weierstrass M-Test (Corollary 6.4.5) gives that $\sum u_n$ converges uniformly on $\mathbf{R}$. Thus h is defined on all of $\mathbf{R}$, with $0 \le h(x) \le 1$.
Monotone: each u_n is nondecreasing. Indeed, let $x < y$. If $u_n(x) = 1/2^n$ then $x > r_n$, hence $y > x > r_n$ and $u_n(y) = 1/2^n$; otherwise $u_n(x) = 0 \le u_n(y)$. Either way $u_n(x) \le u_n(y)$, so for every N

$$\sum_{n=1}^{N} u_n(x) \le \sum_{n=1}^{N} u_n(y),$$

and both sides converge, so the Order Limit Theorem (Theorem 2.3.4) gives $h(x) \le h(y)$. Hence h is monotone increasing on $\mathbf{R}$.
Continuity at each irrational: fix $c \notin \mathbf{Q}$. Each u_n is continuous at every point of $\mathbf{R} \setminus \{r_n\}$, because u_n is constant on the open set $(-\infty, r_n)$ and constant on the open set (r_n, ∞). Since c is irrational, $c \ne r_n$ for every n, so every u_n is continuous at c, and therefore so is each partial sum

$$s_N(x) = \sum_{n=1}^{N} u_n(x)$$

(Theorem 4.3.4, algebraic continuity). Since $s_N \to h$ uniformly on $\mathbf{R}$ and each s_N is continuous at c, the Continuous Limit Theorem (Theorem 6.2.6) gives that h is continuous at c.

Problem 6.5.1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.
(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution. g is defined and continuous exactly on $(-1, 1]$, and $g'(x) = 1/(1 + x)$ on $(-1, 1)$.
Write $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n / n$.
(a) On $(-1, 1)$: for $|x| < 1$ the Ratio Test (Exercise 2.7.9) gives

$$\left| \frac{(-1)^{n+2} x^{n+1}/(n+1)}{(-1)^{n+1} x^n / n} \right| = |x| \, \frac{n}{n+1} \longrightarrow |x| < 1,$$

so the series converges absolutely and g is defined on $(-1, 1)$. Given $x \in (-1, 1)$, choose c with $|x| < c < 1$; the series converges absolutely at c, so by Theorem 6.5.2 it converges uniformly on $[-c, c]$, and a uniform limit of continuous functions is continuous (Theorem 6.4.2). Hence g is continuous at x, and so on all of $(-1, 1)$.
At $x = 1$ the series becomes the alternating harmonic series $\sum (-1)^{n+1}/n$, which converges by the Alternating Series Test (Theorem 2.7.7). So g is defined on $(-1, 1]$. Continuity on $(-1, 1]$: the series converges pointwise on $A = (-1, 1]$, so by Theorem 6.5.5 it converges uniformly on every compact $K \subseteq A$. Any point of A lies inside such a compact subinterval (for $x_0 \in (-1, 1]$ take $K = [-c, 1]$ with $|x_0| \le c < 1$, or $K = [0, 1]$ when $x_0 = 1$), so g is continuous on $(-1, 1]$.

At $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, which diverges (Example 2.4.5). So g is not defined on $[-1, 1]$.
For points $|x| > 1$: no. If the series converged at some x_0 with $|x_0| > 1$, then Theorem 6.5.1 would force absolute convergence at every x with $|x| < |x_0|$, in particular at $x = -1$ — contradicting the divergence just observed.
(b) g is a power series with radius of convergence $R = 1$, so by Theorem 6.5.7 it is differentiable on $(-1, 1)$ with

$$g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1} = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x},$$

the last equality being the geometric series (Example 2.7.5), valid for $|x| < 1$. At $x = 1$ the differentiated series is $1 - 1 + 1 - \cdots$, whose terms do not tend to 0, so it diverges and Theorem 6.5.7 supplies nothing beyond $(-1, 1)$.

Problem 6.5.2. Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.
(a) Converges for every value of $x \in \mathbf{R}$.
(b) Diverges for every value of $x \in \mathbf{R}$.
(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.
(d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.
(e) Converges conditionally at both $x = -1$ and $x = 1$.

Solution. (a) $a_n = 1/n!$. For any x, the Ratio Test (Exercise 2.7.9) gives $|x^{n+1}/(n+1)!| \cdot |n!/x^n| = |x|/(n+1) \to 0 < 1$.
(b) Impossible. Every power series converges at $x = 0$, where the sum is a_0.
(c) $a_n = 1/n^2$ (with $a_0 = 0$), i.e. $\sum_{n=1}^{\infty} x^n/n^2$. For $|x| \leq 1$,

$$\sum_{n=1}^{\infty} \left| \frac{x^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

by the Comparison Test (Theorem 2.7.4) and the p-series with $p = 2$ (Corollary 2.4.7). For $|x| > 1$ we have $|x|^n/n^2 \to \infty$, so the terms do not tend to 0 and the series diverges (Theorem 2.7.3).
(d) Impossible. Absolute convergence at $x = 1$ means $\sum |a_n| < \infty$. But then

$$\sum_{n=0}^{\infty} |a_n(-1)^n| = \sum_{n=0}^{\infty} |a_n| < \infty,$$

so the series converges absolutely at $x = -1$ as well, not conditionally.
(e) Take the even-index series

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{m} x^{2m},$$

that is, $a_{2m} = (-1)^m/m$ for $m \geq 1$ and $a_n = 0$ for all odd n and for $n = 0$. Since x^{2m} takes the value 1 at both $x = 1$ and $x = -1$, the series at either endpoint is the alternating harmonic series $\sum_m (-1)^m/m$, which converges (Alternating Series Test, Theorem 2.7.7), while

$$\sum_{m=1}^{\infty} \left| \frac{(-1)^m}{m} \right| = \sum_{m=1}^{\infty} \frac{1}{m} = \infty.$$

So the convergence is conditional at both endpoints.

Problem 6.5.3. Use the Weierstrass M-Test to prove Theorem 6.5.2, which states: if a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

Solution. Take $M_n = |a_n| c^n$.
For every $x \in [-c, c]$ and every $n \geq 0$,

$$|a_n x^n| = |a_n|\,|x|^n \leq |a_n| c^n = M_n,$$

since $|x| \leq c$. The hypothesis is precisely that

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |a_n|\,|x_0|^n = \sum_{n=0}^{\infty} |a_n x_0^n|$$

converges. Both hypotheses of the Weierstrass M-Test (Corollary 6.4.5) are therefore met by the functions $f_n(x) = a_n x^n$ on the set $[-c, c]$, and the M-Test yields that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$. (Corollary 6.4.5 asks for $M_n > 0$; if some a_n or c itself vanishes, run the test with $M_n + 2^{-n}$ instead, which is still summable.)

6.6 Exercises 6.5.4–6.5.10

Problem 6.5.4. (Term-by-term Antidifferentiation). Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.
(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\, x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.
(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Solution. (a) Fix $x \in (-R, R)$ and choose t with $|x| < t < R$. The original series converges at t, so by Theorem 6.5.1 it converges absolutely at x: $\sum_{n=0}^{\infty} |a_n x^n| < \infty$. Then for every n,

$$\left| \frac{a_n}{n+1} x^{n+1} \right| = \frac{|x|}{n+1}\, |a_n x^n| \leq |x| \cdot |a_n x^n|,$$

so the series defining F converges absolutely by the Comparison Test (Theorem 2.7.4). Hence F is defined on all of $(-R, R)$.
F is itself a power series, $F(x) = \sum_{m=0}^{\infty} b_m x^m$ with $b_0 = 0$ and $b_m = a_{m-1}/m$ for $m \geq 1$, and we have just shown it converges on $(-R, R)$. Theorem 6.5.7 therefore makes F differentiable on $(-R, R)$ with derivative obtained term by term:

$$\begin{aligned} F'(x) &= \sum_{m=1}^{\infty} m\, b_m x^{m-1} = \sum_{m=1}^{\infty} m \cdot \frac{a_{m-1}}{m}\, x^{m-1} \\ &= \sum_{m=1}^{\infty} a_{m-1} x^{m-1} = \sum_{n=0}^{\infty} a_n x^n = f(x). \end{aligned}$$

(b) $g(x) = g(0) + \sum_{n=0}^{\infty} \dfrac{a_n}{n+1} x^{n+1}$.
Indeed $(g - F)'(x) = f(x) - f(x) = 0$ for every $x \in (-R, R)$, so $g - F$ is constant on that interval by Corollary 5.3.4 (a consequence of the Mean Value Theorem). Evaluating at $x = 0$, where $F(0) = 0$, identifies the constant as $g(0)$.

Problem 6.5.5. (a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.
(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6, which states: if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well; consequently the convergence is uniform on compact sets contained in $(-R, R)$.

Solution. (a) Put $x_n = ns^{n-1}$ and fix r with $s < r < 1$. Since

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)s^n}{ns^{n-1}} = s\left(1 + \frac{1}{n}\right) \longrightarrow s < r,$$

there is an $N \in \mathbf{N}$ with $x_{n+1} \leq r\, x_n$ for all $n \geq N$. Induction gives $x_n \leq x_N r^{\,n-N} \leq x_N$ for all $n \geq N$, so

$$ns^{n-1} \leq \max\{x_1, x_2, \ldots, x_N\} =: M \qquad \text{for all } n \geq 1.$$

(b) Fix $x \in (-R, R)$. If $x = 0$ the differentiated series reduces to the single term a_1 and there is nothing to prove, so assume $x \neq 0$ and pick t with $|x| < t < R$. Set $s = |x|/t$, so $0 < s < 1$, and let M be the bound from part (a).
First, $\sum_{n=0}^{\infty} |a_n t^n| < \infty$: choose t' with $t < t' < R$; the series converges at t' by hypothesis, and $|t| < |t'|$, so Theorem 6.5.1 gives absolute convergence at t.
Now estimate the terms of the differentiated series, using $|x|^{n-1} = s^{n-1}t^{n-1}$:

$$\begin{aligned} \left|n a_n x^{n-1}\right| &= \left(ns^{n-1}\right)|a_n|\, t^{n-1} \\ &\leq M\, |a_n|\, t^{n-1} = \frac{M}{t}\, |a_n t^n|\,. \end{aligned}$$

Since $\sum (M/t)|a_n t^n|$ converges, the Comparison Test (Theorem 2.7.4) shows $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges absolutely. As $x \in (-R, R)$ was arbitrary, the differentiated series converges at every point of $(-R, R)$.
For the last clause: $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is itself a power series which we have just shown converges pointwise on the set $A = (-R, R)$. Theorem 6.5.5 then gives uniform convergence on every compact $K \subseteq (-R, R)$.

Problem 6.5.6. Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad \text{for all } |x| < 1.$$

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.

Solution.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2, \qquad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6.$$

The geometric series has radius of convergence $R = 1$, so Theorem 6.5.7 permits term-by-term differentiation on $(-1, 1)$:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \qquad \text{hence} \qquad \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

Multiplying a power series by x shifts the exponents and changes nothing about convergence, so the right-hand series is again a power series valid on $(-1, 1)$. Setting $x = 1/2$:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1/2)^2} = \frac{1/2}{1/4} = 2.$$

Apply Theorem 6.5.7 once more to $\sum_{n=1}^{\infty} nx^n = x/(1-x)^2$ on $(-1,1)$:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right]$$
$$= \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} = \frac{1+x}{(1-x)^3},$$

so $\sum_{n=1}^{\infty} n^2 x^n = x(1+x)/(1-x)^3$. At $x = 1/2$,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{(1/2)^3} = \frac{3/4}{1/8} = 6.$$

Problem 6.5.7. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.
(a) Show that if $L \neq 0$, then the series converges for all x in $(-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful.)
(b) Show that if $L = 0$, then the series converges for all $x \in \mathbf{R}$.
(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \qquad \text{where} \qquad s_n = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\}.$$

(General properties of the limit superior are discussed in Exercise 2.4.7.)

Solution. (a) Fix x with $|x| < 1/L$; for $x = 0$ the series trivially converges to a_0, so take $x \neq 0$ and set $c_n = a_n x^n$, which is nonzero. Then

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \longrightarrow L|x| < 1$$

by the Algebraic Limit Theorem (Theorem 2.3.3). The limiting ratio $L|x|$ is strictly less than 1, which is exactly the hypothesis of the Ratio Test (Exercise 2.7.9), so $\sum c_n = \sum a_n x^n$ converges absolutely.
(b) Same computation. For any $x \neq 0$,

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x| \left|\frac{a_{n+1}}{a_n}\right| \longrightarrow |x| \cdot 0 = 0 < 1,$$

so the Ratio Test gives absolute convergence at every $x \in \mathbf{R}$ (and $x = 0$ is trivial). Equivalently, $L = 0$ means the radius of convergence is $R = \infty$.
(c) One argument covers both, with the convention $1/L' = \infty$ when $L' = 0$. The hypothesis that L' exists forces the ratios $|a_{k+1}/a_k|$ to be bounded, so Exercise 2.4.7(a) applies: $L' = \limsup |a_{n+1}/a_n| \geq 0$, and $L' = L$ whenever the plain limit L exists (Exercise 2.4.7(d)).
Fix $x \neq 0$ with $|x| < 1/L'$, i.e. $L'|x| < 1$, and pick r with $L'|x| < r < 1$. Since $s_n \to L'$ and $r/|x| > L'$, there exists $N \in \mathbf{N}$ with

$$s_N = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq N\right\} < \frac{r}{|x|}.$$

The supremum dominates each of its members, so for every $k \geq N$,

$$\left|\frac{a_{k+1}}{a_k}\right| < \frac{r}{|x|}, \qquad \text{i.e.} \qquad \left|a_{k+1}x^{k+1}\right| < r\left|a_k x^k\right|.$$

Iterating this inequality from $k = N$ gives

$$|a_n x^n| \le \left|a_N x^N\right| r^{n-N} \qquad \text{for all } n \ge N,$$

and $\sum_{n\ge N} |a_N x^N| r^{n-N}$ is a convergent geometric series ($0 < r < 1$). By the Comparison Test (Theorem 2.7.4), $\sum a_n x^n$ converges absolutely.
Thus the series converges on $(-1/L', 1/L')$ when $L' \ne 0$, which is statement (a); and when $L' = 0$ the constraint $L'|x| < 1$ holds for every $x \in \mathbf{R}$, so the series converges on all of $\mathbf{R}$, which is statement (b).

Problem 6.5.8. (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.
(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Solution. (a) The coefficients are read off the derivatives at the origin: $a_k = f^{(k)}(0)/k!$.
Let f denote the common value of the two series on $(-R, R)$. By Theorem 6.5.7 the function f is infinitely differentiable on $(-R, R)$ and every derivative is obtained by term-by-term differentiation of the first series, so for each $k \ge 0$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n x^{n-k}, \qquad x \in (-R, R).$$

Setting $x = 0$ kills every term except $n = k$ (whose factor $x^{n-k} = x^0 = 1$), leaving $f^{(k)}(0) = k!\, a_k$. Running the identical computation on $\sum b_n x^n$, which represents the same function f, gives $f^{(k)}(0) = k!\, b_k$. Hence $k!\, a_k = k!\, b_k$, and $a_k = b_k$ for all $k = 0, 1, 2, \ldots$.
(b) $a_n = 1/n!$ for every $n \ge 0$.
From $f(0) = 1$ we get $a_0 = 1$. By Theorem 6.5.7 again,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

a power series converging on all of $(-R, R)$. The hypothesis $f' = f$ makes this the same function as $\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$, so part (a) applies and matches coefficients:

$$(n+1)a_{n+1} = a_n \quad \text{for all } n \ge 0, \qquad \text{i.e.} \qquad a_{n+1} = \frac{a_n}{n+1}.$$

With $a_0 = 1$ this recursion gives $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/6$, and by induction

$$a_n = \frac{1}{n!}, \qquad n = 0, 1, 2, \ldots$$

Problem 6.5.9. Review the definitions and results from Section 2.8 concerning products of series and Cauchy products in particular. At the end of Section 2.9, we mentioned the following result: If both $\sum a_n$ and $\sum b_n$ converge conditionally to A and B respectively, then it is possible for the Cauchy product,

$$\sum d_n \quad \text{where} \quad d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

to diverge. However, if $\sum d_n$ does converge, then it must converge to AB. To prove this, set

$$f(x) = \sum a_n x^n, \qquad g(x) = \sum b_n x^n, \qquad h(x) = \sum d_n x^n.$$

Use Abel's Theorem and the result in Exercise 2.8.7 to establish this result.

Solution. The whole proof is the identity $h(x) = f(x)g(x)$ on $(-1,1)$ followed by $x \to 1^-$.
Assume $\sum a_n = A$, $\sum b_n = B$, and $\sum d_n = D$ all converge. Each of the three power series therefore converges at $x = 1$, so Abel's Theorem (Theorem 6.5.4, with $R = 1$) gives uniform convergence on $[0,1]$; the partial sums are polynomials, hence by the Term-by-term Continuity Theorem (Theorem 6.4.2) f, g, h are continuous on $[0,1]$. In particular

$$\lim_{x\to 1^-} f(x) = A, \qquad \lim_{x\to 1^-} g(x) = B, \qquad \lim_{x\to 1^-} h(x) = D.$$

Now fix x with $|x| < 1$. Since $\sum a_n \cdot 1^n$ converges, Theorem 6.5.1 says $\sum a_n x^n$ converges absolutely; likewise $\sum b_n x^n$. So Exercise 2.8.7 applies to the two absolutely convergent series $\sum (a_n x^n)$ and $\sum (b_n x^n)$: their Cauchy product converges to the product of the sums. The n-th Cauchy term of these two series is

$$\sum_{k=0}^{n} (a_k x^k)(b_{n-k} x^{n-k}) = \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big) x^n = d_n x^n,$$

so Exercise 2.8.7 reads

$$h(x) = \sum_{n=0}^{\infty} d_n x^n = f(x)g(x) \qquad \text{for all } |x| < 1.$$

(The conclusion of 2.8.7 includes the convergence of the Cauchy product, so h needs no separate convergence check on $(-1,1)$.)
Let $x \to 1^-$ in this identity: the left side tends to D and the right side to AB, by the three limits above and Corollary 4.2.4. Hence

$$\sum_{n=0}^{\infty} d_n = D = AB.$$

Problem 6.5.10. Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R,R)$, and assume $(x_n) \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbf{N}$, show that $g(x)$ must be identically zero on all of $(-R,R)$.

Solution. Factor out the lowest surviving power: $g(x) = x^m h(x)$ with h continuous and $h(0) = b_m$, and the vanishing along (x_n) forces $h(0) = 0$.
If every $b_n = 0$ there is nothing to prove, so assume some coefficient is nonzero and let

$$m = \min\{\, n \geq 0 : b_n \neq 0 \,\}.$$

Define

$$h(x) = \sum_{k=0}^{\infty} b_{m+k}\, x^k.$$

This power series converges on $(-R,R)$: for $0 < |x| < R$ the series $\sum_k b_{m+k} x^{m+k}$ is a tail of the convergent series $\sum_n b_n x^n$, and dividing its partial sums by the nonzero constant x^m shows $\sum_k b_{m+k} x^k$ converges; at $x = 0$ convergence is trivial. Moreover, since $b_0 = \cdots = b_{m-1} = 0$,

$$g(x) = \sum_{n=m}^{\infty} b_n x^n = x^m h(x) \qquad \text{for all } x \in (-R,R).$$

By Theorem 6.5.7, h is continuous on $(-R,R)$. For each n we have $x_n^m\, h(x_n) = g(x_n) = 0$ with $x_n \neq 0$, so $h(x_n) = 0$. Continuity of h at 0 together with $x_n \to 0$ (Theorem 4.3.2, the sequential characterization of continuity) gives

$$b_m = h(0) = \lim_{n\to\infty} h(x_n) = 0,$$

contradicting $b_m \neq 0$. Hence every $b_n = 0$ and $g \equiv 0$ on $(-R, R)$.

6.7 Exercises 6.5.11–6.6.6

Problem 6.5.11. A series $\sum_{n=0}^{\infty} a_n$ is said to be Abel-summable to L if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0, 1)$ and $L = \lim_{x\to 1^-} f(x)$.
(a) Show that any series that converges to a limit L is also Abel-summable to L.
(b) Show that $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and find the sum.

Solution. (a) Abel's Theorem makes f continuous at the endpoint, so $\lim_{x\to 1^-} f(x) = f(1) = L$.
Assume $\sum_{n=0}^{\infty} a_n = L$; that is, $f(x) = \sum a_n x^n$ converges at $x = 1$ with value L. Theorem 6.5.1 then gives absolute convergence for every $|x| < 1$, so f is defined on all of $[0, 1]$. Abel's Theorem (Theorem 6.5.4 with $R = 1$, whose hypothesis "converges at $x = R$" is exactly our assumption) makes the convergence uniform on $[0, 1]$, and the partial sums are polynomials, so the Term-by-term Continuity Theorem (Theorem 6.4.2) makes f continuous on $[0, 1]$. Continuity at $x = 1$ gives

$$\lim_{x\to 1^-} f(x) = f(1) = \sum_{n=0}^{\infty} a_n = L.$$

Thus $\sum a_n$ is Abel-summable to L.
(b) The sum is $1/2$.
Here $a_n = (-1)^n$, and for $x \in [0, 1)$ the series $\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n$ is geometric with ratio $-x$ satisfying $|-x| = x < 1$, so by Example 2.7.5 it converges with

$$f(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

Since $x \mapsto 1/(1 + x)$ is continuous at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2},$$

so $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable to $1/2$.

Problem 6.6.1. The derivation in Example 6.6.1 shows the Taylor series for $\arctan(x)$ is valid for all $x \in (-1, 1)$. Notice, however, that the series also converges when $x = 1$. Assuming that $\arctan(x)$ is continuous, explain why the value of the series at $x = 1$ must necessarily be $\arctan(1)$. What interesting identity do we get in this case?

Solution. The value is $\arctan(1) = \pi/4$, and the identity is Leibniz's series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Let $S(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$. At $x = 1$ it reads $\sum(-1)^n/(2n+1)$, whose terms decrease monotonically to zero, so the Alternating Series Test (Theorem 2.7.7) applies and S converges on a set $A \supseteq (-1, 1]$. Theorem 6.5.7 makes S continuous on all of A, the endpoint $x = 1$ included.

Since $S(x) = \arctan(x)$ on $(-1, 1)$ by equation (2) of the section, and arctan is continuous at 1 by hypothesis, letting $x \to 1^-$ gives

$$\begin{aligned} S(1) &= \lim_{x \to 1^-} S(x) = \lim_{x \to 1^-} \arctan(x) \\ &= \arctan(1) = \frac{\pi}{4}. \end{aligned}$$

Writing out $S(1)$ gives the stated identity.

Problem 6.6.2. Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?
(a) $x \cos(x^2)$
(b) $x/(1 + 4x^2)^2$
(c) $\log(1 + x^2)$

Solution. (a) $x \cos(x^2) = \sum_{n=0}^{\infty} (-1)^n x^{4n+1}/(2n)!$, valid for all $x \in \mathbf{R}$.
The series $x - x^3/3! + x^5/5! - \cdots$ converges uniformly to $\sin(x)$ on every $[-R, R]$ (Example 6.6.4), so Theorem 6.5.7 permits term-by-term differentiation on all of $\mathbf{R}$:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad x \in \mathbf{R}.$$

Substituting x^2 for x is legitimate at every real x since $x^2 \in \mathbf{R}$, and pulling the constant x through the sum is Theorem 2.7.1:

$$\begin{aligned} x \cos(x^2) &= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+1}. \end{aligned}$$

(b) $\dfrac{x}{(1 + 4x^2)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) 4^n x^{2n+1}$, valid precisely for $|x| < 1/2$.
Example 6.6.1 gives $1/(1 - x)^2 = \sum_{n=0}^{\infty} (n+1) x^n$ for $|x| < 1$. Substituting $-4x^2$ for x, which requires $|-4x^2| < 1$, i.e. $|x| < 1/2$,

$$\frac{1}{(1 + 4x^2)^2} = \sum_{n=0}^{\infty} (n+1)(-4)^n x^{2n},$$

and multiplying by x gives the displayed series. At $|x| = 1/2$ the nth term has absolute value $(n + 1)4^n 2^{-(2n+1)} = (n+1)/2$, which does not tend to zero, so the series diverges there and hence for all $|x| \geq 1/2$.
(c) $\log(1 + x^2) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} x^{2n}$, valid precisely for $|x| \leq 1$.
Replacing x by $-x$ in equation (1) gives $1/(1 + x) = \sum_{n=0}^{\infty} (-1)^n x^n$ on $(-1, 1)$; term-by-term antidifferentiation (Exercise 6.5.4) produces F on $(-1, 1)$ with $F' = 1/(1 + x)$ and $F(0) = 0$, and $\log(1 + x)$ has the same derivative and the same value at 0, so the two agree by Corollary 5.3.4:

$$\log(1 + u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n, \qquad u \in (-1, 1),$$

and Exercise 6.6.4 extends this to $u = 1$. Since $u = x^2$ ranges over $[0, 1]$ exactly when $|x| \leq 1$, the substitution is legitimate there and gives the stated series. For $|x| > 1$ the terms x^{2n}/n are unbounded, so the series diverges (Theorem 2.7.3).

Problem 6.6.3. Derive the formula for the Taylor coefficients given in Theorem 6.6.2. (That theorem asserts that if

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

is defined on some nontrivial interval centered at zero, then $a_n = f^{(n)}(0)/n!$.)

Solution. Differentiate the series k times and set $x = 0$: the terms below x^k have been annihilated and those above still carry a factor of x, so only $k!\,a_k$ survives.
The interval of convergence contains some $(-R, R)$ with $R > 0$, and on it Theorem 6.5.7 makes f infinitely differentiable with every derivative computed term by term. Applying that k times gives, for $x \in (-R, R)$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,a_nx^{n-k},$$

the terms with $n < k$ having been annihilated one at a time. (Formally: the case $k = 0$ is the hypothesis, and differentiating the displayed formula for k term by term multiplies a_nx^{n-k} by $(n-k)$ and lowers the exponent, which is exactly the formula for $k+1$.)
Evaluate at $x = 0$. Every summand with $n > k$ carries a factor x^{n-k} with $n - k \geq 1$ and so vanishes, while the $n = k$ summand is the constant $k(k-1)\cdots 1 \cdot a_k = k!\,a_k$. Hence

$$f^{(k)}(0) = k!\,a_k, \qquad \text{i.e.} \qquad a_k = \frac{f^{(k)}(0)}{k!},$$

for every $k = 0, 1, 2, \ldots$ (with $0! = 1$ and $f^{(0)} = f$, which gives $a_0 = f(0)$).

Problem 6.6.4. Explain how Lagrange's Remainder Theorem can be modified to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log(2).$$

Solution. The modification is to allow x to be an endpoint. Theorem 6.6.3 is stated for x in the open interval $(-R, R)$, but its proof only ever applies the Generalized Mean Value Theorem (Theorem 5.3.5) to E_N and x^{N+1} on the closed interval $[0, x]$ and its successive subintervals, so it holds verbatim in the form: if f is $N+1$ times differentiable on an open interval containing $[0, b]$, $a_n = f^{(n)}(0)/n!$ for $0 \leq n \leq N$, and $S_N(x) = a_0 + a_1x + \cdots + a_Nx^N$, then each $x \in (0, b]$ admits $c \in (0, x)$ with

$$E_N(x) = f(x) - S_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

Apply this to $f(x) = \log(1 + x)$ with $b = 1$ and $x = 1$; the hypothesis holds because f is infinitely differentiable on $(-1, \infty) \supseteq [0, 1]$, with

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad (n \geq 1),$$

so $a_0 = 0$ and $a_n = f^{(n)}(0)/n! = (-1)^{n-1}/n$, making

$$S_N(1) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{N-1}}{N}$$

the Nth partial sum of the series in question. For the error there is $c \in (0, 1)$ with

$$\begin{aligned} E_N(1) &= \frac{f^{(N+1)}(c)}{(N+1)!} \cdot 1^{N+1} \\ &= \frac{(-1)^N N!}{(1+c)^{N+1}(N+1)!} = \frac{(-1)^N}{(N+1)(1+c)^{N+1}}. \end{aligned}$$

Since $c > 0$ we have $(1+c)^{N+1} > 1$, whence

$$|E_N(1)| \le \frac{1}{N+1} \longrightarrow 0.$$

Therefore $S_N(1) \to f(1) = \log 2$, which is the asserted identity.

Problem 6.6.5. (a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.
(b) Verify the formula $f'(x) = e^x$.
(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Solution. (a) $a_n = 1/n!$, so the Taylor series is $\sum_{n=0}^{\infty} x^n/n!$.
Indeed $f^{(n)}(x) = e^x$ for every n, so $a_n = f^{(n)}(0)/n! = e^0/n! = 1/n!$ by Theorem 6.6.2.
Fix $R > 0$. Since f is infinitely differentiable on all of $\mathbf{R}$, Lagrange's Remainder Theorem (Theorem 6.6.3) applies on the interval $(-(R+1), R+1)$, which contains $[-R, R]$: for each $x \in [-R, R]$ with $x \neq 0$ there is c with $|c| < |x| \le R$ and

$$E_N(x) = e^x - S_N(x) = \frac{e^c}{(N+1)!} x^{N+1}.$$

Because $e^c \le e^R$ and $|x| \le R$, and because $E_N(0) = 0$,

$$\sup_{x \in [-R,R]} |E_N(x)| \le \frac{e^R R^{N+1}}{(N+1)!}.$$

The series $\sum R^n/n!$ converges by the Ratio Test (Exercise 2.7.9), since the ratio of consecutive terms is $R/(n+1) \to 0$; hence its terms satisfy $R^{N+1}/(N+1)! \to 0$ (Theorem 2.7.3). The right-hand bound is a constant multiple of that, so it tends to 0, and $S_N \to e^x$ uniformly on $[-R, R]$.
(b) Every $x \in \mathbf{R}$ lies in some $[-R, R]$, so by (a) the series $\sum x^n/n!$ converges to e^x on all of $\mathbf{R}$. Theorem 6.5.7 then licenses term-by-term differentiation:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x. \end{aligned}$$

(c) Substituting $-x$ for x (valid for all $x \in \mathbf{R}$ since $-x \in \mathbf{R}$),

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n.$$

Collecting the coefficient of x^n in the product of the two series means summing $\big(1/k!\big)\big((-1)^{n-k}/(n-k)!\big)$ over $0 \le k \le n$:

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!} &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} \\ &= \frac{(1-1)^n}{n!} = 0 \qquad (n \ge 1), \end{aligned}$$

by the binomial theorem, while for $n = 0$ the coefficient is 1. So the product series is the constant 1, in agreement with $e^x \cdot e^{-x} = 1$.

Problem 6.6.6. Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

introduced at the end of this section.
(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find $g''(0)$.
(b) Compute $g''(x)$ and $g'''(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)}(x)$ at points different from zero.
(c) Construct a general argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbf{N}$.

Solution. One limit settles all three parts:

$$(*) \qquad \lim_{x\to 0} q(1/x)\, e^{-1/x^2} = 0 \quad \text{for every polynomial } q.$$

Put $t = 1/x$, so $|t| \to \infty$; by the Algebraic Limit Theorem it suffices to show $|t|^m e^{-t^2} \to 0$ for each fixed $m \in \mathbf{N}$. For $|t| \geq 1$ we have $|t|^m \leq t^{2m} = s^m$ with $s = t^2 \to \infty$, and the series of Exercise 6.6.5(a) has all positive terms for $s > 0$, so $e^s > s^{m+1}/(m+1)!$ and therefore

$$0 \leq \frac{|t|^m}{e^{t^2}} \leq \frac{s^m}{e^s} < \frac{(m+1)!}{s} \longrightarrow 0.$$

(a) For $x \neq 0$ the chain rule gives

$$g'(x) = e^{-1/x^2} \cdot \frac{d}{dx}\Big(-\frac{1}{x^2}\Big) = \frac{2}{x^3} e^{-1/x^2}.$$

Since $g'(0) = 0$ was computed in the text, the definition of the derivative gives

$$\begin{aligned} g''(0) &= \lim_{x\to 0} \frac{g'(x) - g'(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{2}{x^4} e^{-1/x^2} = 0 \end{aligned}$$

by $(*)$ with $q(t) = 2t^4$.
(b) Differentiating again for $x \neq 0$,

$$\begin{aligned} g''(x) &= \Big(\frac{4}{x^6} - \frac{6}{x^4}\Big) e^{-1/x^2}, \\ g'''(x) &= \Big(\frac{8}{x^9} - \frac{36}{x^7} + \frac{24}{x^5}\Big) e^{-1/x^2}. \end{aligned}$$

The pattern is that each derivative is a polynomial in $1/x$ times e^{-1/x^2}. Precisely: there are polynomials p_n with

$$g^{(n)}(x) = p_n(1/x)\, e^{-1/x^2} \qquad (x \neq 0),$$

defined recursively by $p_0(t) = 1$ and

$$p_{n+1}(t) = -t^2 p_n'(t) + 2t^3 p_n(t).$$

Indeed, differentiating $p_n(1/x)e^{-1/x^2}$ by the product and chain rules gives

$$\Big[p_n'(1/x)\Big(-\frac{1}{x^2}\Big) + p_n(1/x)\frac{2}{x^3}\Big] e^{-1/x^2},$$

which is $p_{n+1}(1/x)e^{-1/x^2}$; and p_{n+1} is again a polynomial. (The recursion returns $p_1(t) = 2t^3$, $p_2(t) = 4t^6 - 6t^4$, $p_3(t) = 8t^9 - 36t^7 + 24t^5$, matching the computations above.)
(c) Induct on n, the induction hypothesis being that $g^{(n)}$ exists on all of $\mathbf{R}$, that $g^{(n)}(0) = 0$, and that $g^{(n)}(x) = p_n(1/x)e^{-1/x^2}$ for $x \neq 0$. The case $n = 0$ is the definition of g (with $p_0 = 1$). Assuming it for

n, part (b) supplies the formula for $g^{(n+1)}$ away from zero, and at zero the definition of the derivative gives

$$\begin{aligned} g^{(n+1)}(0) &= \lim_{x\to 0} \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{1}{x}\, p_n(1/x)\, e^{-1/x^2} = 0 \end{aligned}$$

by $(*)$ applied to the polynomial $q(t) = t\,p_n(t)$. Thus $g^{(n+1)}$ exists everywhere and vanishes at the origin, completing the induction: g is infinitely differentiable with $g^{(n)}(0) = 0$ for all n.

6.8 Exercises 6.6.7–6.7.3

Problem 6.6.7. Find an example of each of the following or explain why no such function exists.
(a) An infinitely differentiable function $g(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.
(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.
(c) An infinitely differentiable function $f(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Solution. All three exist.
(a) $g(x) = \dfrac{1}{1+x^2}$.
It is infinitely differentiable on $\mathbf{R}$ (the denominator never vanishes), and Example 6.6.1 substitutes $-x^2$ into equation (1) to give

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } |x| < 1.$$

By Theorem 6.6.2 this power series is precisely the Taylor series of g. For $|x| \geq 1$ the terms satisfy $|(-1)^n x^{2n}| \geq 1$, so they do not tend to zero and the series diverges (Theorem 2.7.3). Hence the Taylor series converges to g exactly on $(-1, 1)$.
(b) $h(x) = \sin(x) + g(x)$, where g is the flat function of Exercise 6.6.6.
By Exercise 6.6.6(c), $g^{(n)}(0) = 0$ for every n, so

$$h^{(n)}(0) = \sin^{(n)}(0) + g^{(n)}(0) = \sin^{(n)}(0),$$

and h has exactly the Taylor series $x - x^3/3! + x^5/5! - \cdots$ of $\sin(x)$. Yet for $x \neq 0$,

$$h(x) - \sin(x) = e^{-1/x^2} > 0,$$

so $h(x) \neq \sin(x)$ at every nonzero point. (h is infinitely differentiable as a sum of two such functions.)
(c) Take

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

On $(0, \infty)$ and on $(-\infty, 0)$ the function is visibly infinitely differentiable, with

$$f^{(n)}(x) = p_n(1/x)e^{-1/x^2} \ (x > 0), \qquad f^{(n)}(x) = 0 \ (x < 0),$$

for the polynomials p_n of Exercise 6.6.6(b). At the origin, induction as in Exercise 6.6.6(c) gives $f^{(n)}(0) = 0$ for all n: the case $n = 0$ is $f(0) = 0$, and assuming it for n, the left-hand difference quotient of $f^{(n)}$ at 0 is identically 0, while the right-hand quotient is $(1/x)p_n(1/x)e^{-1/x^2} \to 0$ by limit $(*)$ of Exercise 6.6.6; the two one-sided limits agree, so $f^{(n+1)}(0) = 0$.
Consequently every Taylor coefficient of f is 0, so the Taylor series is the zero series, which converges for all $x \in \mathbf{R}$ with sum 0. Its sum equals $f(x)$ precisely when $f(x) = 0$, that is, precisely when $x \leq 0$.

Problem 6.6.8. Here is a weaker form of Lagrange's Remainder Theorem whose proof is arguably more illuminating than the one for the stronger result.
(a) First establish a lemma: If g and h are differentiable on $[0,x]$ with $g(0)=h(0)$ and $g'(t)\le h'(t)$ for all $t\in[0,x]$, then $g(t)\le h(t)$ for all $t\in[0,x]$.
(b) Let f, S_N, and E_N be as in Theorem 6.6.3, and take $0<x<R$. If $|f^{(N+1)}(t)|\le M$ for all $t\in[0,x]$, show

$$|E_N(x)|\le \frac{Mx^{N+1}}{(N+1)!}.$$

Solution. (a) Apply the Mean Value Theorem (Theorem 5.3.2) to $\varphi=h-g$. For $t\in(0,x]$ there is $d\in(0,t)$ with

$$\varphi(t)=\varphi(t)-\varphi(0)=\varphi'(d)\,t=\big(h'(d)-g'(d)\big)t\ge 0,$$

since $g'(d)\le h'(d)$ and $t>0$; and $\varphi(0)=0$. Hence $g(t)\le h(t)$ on $[0,x]$.
(b) Because S_N is a polynomial of degree at most N matching the first N derivatives of f at 0 (as in the proof of Theorem 6.6.3),

$$E_N^{(n)}(0)=0 \quad (0\le n\le N), \qquad E_N^{(N+1)}(t)=f^{(N+1)}(t).$$

Claim: for each $k=0,1,\dots,N+1$,

$$\big|E_N^{(N+1-k)}(t)\big|\le \frac{Mt^k}{k!} \quad \text{for all } t\in[0,x].$$

The case $k=0$ is the hypothesis $|f^{(N+1)}(t)|\le M$. Assume the claim for some $k\le N$ and write $m=N-k$, so $0\le m\le N$ and $E_N^{(m)}(0)=0$. Two applications of the lemma on $[0,x]$, with the pairs

$$g(t)=E_N^{(m)}(t), \qquad h(t)=\frac{Mt^{k+1}}{(k+1)!},$$
$$g(t)=\frac{-Mt^{k+1}}{(k+1)!}, \quad h(t)=E_N^{(m)}(t),$$

are legitimate: both pairs agree at $t=0$ (all four values are 0), and in each case the derivative inequality is exactly the inductive hypothesis

$$-\frac{Mt^k}{k!}\le E_N^{(m+1)}(t)\le \frac{Mt^k}{k!}.$$

The two conclusions combine to give $|E_N^{(m)}(t)|\le Mt^{k+1}/(k+1)!$ on $[0,x]$, which is the claim for $k+1$. Taking $k=N+1$ (so $N+1-k=0$) and then $t=x$:

$$|E_N(x)|\le \frac{Mx^{N+1}}{(N+1)!}.$$

Problem 6.6.9. (Cauchy's Remainder Theorem). Let f be differentiable $N+1$ times on $(-R,R)$. For each $a\in(-R,R)$, let $S_N(x,a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x,a)=\sum_{n=0}^{N} c_n(x-a)^n \quad \text{where} \quad c_n=\frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x,a)=f(x)-S_N(x,a)$. Now fix $x\neq 0$ in $(-R,R)$ and consider $E_N(x,a)$ as a function of a.
(a) Find $E_N(x,x)$.

(b) Explain why $E_N(x,a)$ is differentiable with respect to a, and show

$$E_N'(x,a) = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Show

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Solution. (a) $E_N(x,x) = 0$: every term of $S_N(x,x)$ with $n \geq 1$ carries the factor $(x-x)^n = 0$, so $S_N(x,x) = c_0 = f(x)$.

(b) Differentiability: f is $N+1$ times differentiable on $(-R,R)$, so each $f^{(n)}$ with $n \leq N$ is differentiable there, and $a \mapsto (x-a)^n$ is a polynomial; $S_N(x,\cdot)$ is a finite sum of products of differentiable functions, and $f(x)$ is a constant in a. Differentiating term by term with the product rule,

$$\frac{d}{da}\left[\frac{f^{(n)}(a)}{n!}(x-a)^n\right] = \frac{f^{(n+1)}(a)}{n!}(x-a)^n - \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1},$$

the second term being absent when $n=0$. Writing $A_n = f^{(n+1)}(a)(x-a)^n/n!$, the subtracted term for index n is exactly A_{n-1}, so the sum telescopes:

$$\begin{aligned}\frac{\partial}{\partial a}S_N(x,a) &= \sum_{n=0}^{N}\big(A_n - A_{n-1}\big)\\ &= A_N = \frac{f^{(N+1)}(a)}{N!}(x-a)^N.\end{aligned}$$

Hence

$$E_N'(x,a) = -\frac{\partial}{\partial a}S_N(x,a) = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N.$$

(c) Apply the Mean Value Theorem (Theorem 5.3.2) to $a \mapsto E_N(x,a)$ on the closed interval with endpoints 0 and x; by (b) this function is differentiable on all of $(-R,R)$, hence continuous on that closed interval and differentiable on its interior. So there is c between 0 and x with

$$\begin{aligned}E_N(x,x) - E_N(x,0) &= E_N'(x,c)\,(x-0)\\ &= \frac{-f^{(N+1)}(c)}{N!}(x-c)^N x.\end{aligned}$$

Since $E_N(x,x) = 0$ by (a),

$$E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x.$$

Problem 6.6.10. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0,1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem proved in Exercise 6.6.9 to show the series representation for f holds on $[0,1)$.

Solution. The series is

$$\sum_{n=0}^{\infty} c_n x^n, \qquad c_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = \frac{1}{4^n}\binom{2n}{n},$$

with $c_0 = 1$; an induction on n gives

$$f^{(n)}(t) = \frac{1\cdot 3\cdots(2n-1)}{2^n}\,(1-t)^{-(2n+1)/2}$$

(differentiate once; each step multiplies by $(2n+1)/2$ and lowers the exponent by 1), whence $c_n = f^{(n)}(0)/n! = 1\cdot 3\cdots(2n-1)/(2^n n!)$ and $2^n n! = 2\cdot 4\cdots(2n)$.
Two facts used throughout. First, $0 < c_n \le 1$, and in fact

$$c_n^2 \le \prod_{k=1}^{n} \frac{2k-1}{2k}\cdot\frac{2k}{2k+1} = \frac{1}{2n+1}, \qquad \text{so} \qquad c_n \le \frac{1}{\sqrt{2n+1}},$$

because $\frac{2k-1}{2k} < \frac{2k}{2k+1}$ for every k. Second,

$$\frac{f^{(N+1)}(t)}{(N+1)!} = c_{N+1}\,(1-t)^{-(2N+3)/2}.$$

(a) Fix $0 < x \le 1/2$ (the case $x = 0$ being trivial). Lagrange's Remainder Theorem (Theorem 6.6.3) applies, since f is infinitely differentiable on $(-1,1)$ and x lies in $(-R,R)$ for $R = 3/4$; it gives c with $|c| < x$, so

$$|E_N(x)| = c_{N+1}\,\frac{x^{N+1}}{(1-c)^{N+1}\,(1-c)^{1/2}} \le c_{N+1}\,\frac{1}{\sqrt{1-x}}\left(\frac{x}{1-x}\right)^{N+1},$$

using only $0 < 1-x < 1-c$, which follows from $c < x$.
(i) $0 < x < 1/2$. Then $r = x/(1-x) < 1$, so $|E_N(x)| \le (1-x)^{-1/2} r^{N+1} \to 0$ (Example 2.5.3), using $c_{N+1} \le 1$.
(ii) $x = 1/2$. Now $r = 1$ and the bound degenerates to the constant $\sqrt{2}\,c_{N+1}$; the extra care is supplied by $c_{N+1} \le 1/\sqrt{2N+3}$:

$$|E_N(1/2)| \le \sqrt{2}\,c_{N+1} \le \frac{\sqrt{2}}{\sqrt{2N+3}} \longrightarrow 0.$$

In both cases $E_N(x) \to 0$, so $S_N(x) \to f(x)$ on $[0,1/2]$.
For $1/2 < x < 1$ the attempt fails. Theorem 6.6.3 pins c down no further than $|c| < x$, and the resulting uniform estimate is the one displayed above, whose factor $(x/(1-x))^{N+1}$ grows geometrically since $x/(1-x) > 1$, while c_{N+1} decays only like $N^{-1/2}$. The bound therefore tends to ∞ rather than 0 and yields no information.
(b) Fix $0 < x < 1$ ($x = 0$ again trivial). Cauchy's Remainder Theorem (Exercise 6.6.9, applied on $(-R,R)$ with $x < R < 1$, where f is $N+1$ times differentiable) gives c between 0 and x with

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x = \frac{2N+1}{2}\,c_N\,\frac{(x-c)^N x}{(1-c)^{N+3/2}},$$

since $f^{(N+1)}(c)/N! = (N+1)c_{N+1}(1-c)^{-(2N+3)/2}$ and $(N+1)c_{N+1} = \frac{2N+1}{2}c_N$.
The decisive estimate is that for $0 < c < x < 1$,

$$\frac{x-c}{1-c} \le x \iff x - c \le x - xc \iff xc \le c,$$

which holds because $x < 1$ and $c > 0$. Combining this with $(1-c)^{-3/2} \le (1-x)^{-3/2}$ and $c_N \le 1/\sqrt{2N+1}$:

$$|E_N(x)| \le \frac{2N+1}{2} c_N\,\frac{x^{N+1}}{(1-x)^{3/2}} \le \frac{\sqrt{2N+1}}{2}\cdot\frac{x^{N+1}}{(1-x)^{3/2}}.$$

The right side tends to 0, since x^{N+1} decays geometrically while $\sqrt{2N+1}$ grows only polynomially: writing $t_N = \sqrt{2N+1}\,x^{N+1}$ we have $t_{N+1}/t_N \to x < 1$, so $\sum t_N$ converges by the Ratio Test (Exercise 2.7.9) and in particular $t_N \to 0$ (Theorem 2.7.3). Hence $E_N(x) \to 0$ and the series represents f on all of $[0,1)$.

Problem 6.7.1. Assuming WAT, show that if f is continuous on $[a,b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a,b]$.
(Here WAT is the Weierstrass Approximation Theorem, Theorem 6.7.1: if $f : [a,b] \to \mathbf{R}$ is continuous and $\epsilon > 0$, then there is a polynomial $p(x)$ with $|f(x) - p(x)| < \epsilon$ for all $x \in [a,b]$.)

Solution. Run WAT once for each tolerance $\epsilon = 1/n$.
For each $n \in \mathbf{N}$, apply Theorem 6.7.1 to the continuous function f with $\epsilon = 1/n$; this produces a polynomial p_n with

$$|f(x) - p_n(x)| < \frac{1}{n} \quad \text{for all } x \in [a,b].$$

Given $\epsilon > 0$, choose $N \in \mathbf{N}$ with $N > 1/\epsilon$ (Archimedean Property, Theorem 1.4.2). Then for every $n \geq N$ and every $x \in [a,b]$,

$$|f(x) - p_n(x)| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Since N depends on ϵ alone and not on x, this is precisely Definition 6.2.3: $p_n \to f$ uniformly on $[a,b]$.

Problem 6.7.2. Prove Theorem 6.7.3.
(Theorem 6.7.3: Let $f : [a,b] \to \mathbf{R}$ be continuous. Given $\epsilon > 0$, there exists a polygonal function ϕ satisfying $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a,b]$. Recall Definition 6.7.2: a continuous $\phi : [a,b] \to \mathbf{R}$ is polygonal if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that ϕ is linear on each subinterval $[x_{i-1}, x_i]$, $i = 1, \ldots, n$.)

Solution. Take ϕ to be the polygonal interpolation of f at the nodes of any partition of mesh smaller than a uniform-continuity δ for $\epsilon/2$.
Since $[a,b]$ is compact (Theorem 3.3.4) and f is continuous, f is uniformly continuous on $[a,b]$ by Theorem 4.4.7: there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2} \qquad (x, y \in [a,b]).$$

Fix $n \in \mathbf{N}$ with $(b-a)/n < \delta$ and let $x_i = a + i(b-a)/n$ for $i = 0, 1, \ldots, n$, so that $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of mesh $(b-a)/n < \delta$. Define ϕ on each $[x_{i-1}, x_i]$ by

$$\phi(x) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x - x_{i-1}).$$

On each subinterval this is a linear function, and the two formulas attached to adjacent subintervals agree at the shared endpoint (both give $f(x_i)$), so ϕ is well defined and continuous on $[a,b]$; it is polygonal in the sense of Definition 6.7.2.
Now fix $x \in [a,b]$ and choose i with $x \in [x_{i-1}, x_i]$. Writing $t = (x - x_{i-1})/(x_i - x_{i-1}) \in [0,1]$, the displayed formula says exactly that $\phi(x)$ is the convex combination

$$\phi(x) = (1-t) f(x_{i-1}) + t\, f(x_i).$$

Both $|x - x_{i-1}|$ and $|x - x_i|$ are at most $x_i - x_{i-1} < \delta$, so

$$\begin{aligned} |f(x) - \phi(x)| &= \big|(1-t)\big(f(x) - f(x_{i-1})\big) + t\big(f(x) - f(x_i)\big)\big| \\ &\leq (1-t)\,|f(x) - f(x_{i-1})| + t\,|f(x) - f(x_i)| \\ &\leq (1-t)\frac{\epsilon}{2} + t\,\frac{\epsilon}{2} \;=\; \frac{\epsilon}{2} \;<\; \epsilon. \end{aligned}$$

Since $x \in [a,b]$ was arbitrary, ϕ is the required polygonal function.

Problem 6.7.3. (a) Find the second degree polynomial $p(x) = q_0 + q_1 x + q_2 x^2$ that interpolates the three points $(-1, 1)$, $(0, 0)$, and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.
(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the points $x = -1, -1/2, 0, 1/2,$ and 1. Add a sketch of this polynomial to the graph from (a).

Solution. (a) $p(x) = x^2$.
The three interpolation conditions read

$$\begin{aligned} p(0) &= q_0 = 0, \\ p(1) &= q_0 + q_1 + q_2 = 1, \\ p(-1) &= q_0 - q_1 + q_2 = 1. \end{aligned}$$

Subtracting the third equation from the second gives $2q_1 = 0$, so $q_1 = 0$ and then $q_2 = 1$.
Sketch: $g(x) = |x|$ is the V with vertex at the origin and endpoints $(\pm 1, 1)$; $p(x) = x^2$ is the parabola meeting it exactly at $x = -1, 0, 1$ and lying strictly below it in between (since $x^2 < |x|$ for $0 < |x| < 1$), sagging to a maximum gap of $|x| - x^2 = 1/4$ at $x = \pm 1/2$.
(b) $p(x) = \dfrac{7}{3}x^2 - \dfrac{4}{3}x^4$.
The five nodes are symmetric about 0 and g is even, so the interpolating polynomial of degree at most four is even as well: if p interpolates the data then so does $x \mapsto p(-x)$, and the interpolant through five points is unique among polynomials of degree ≤ 4, forcing $p(-x) = p(x)$. Write $p(x) = b_0 + b_2 x^2 + b_4 x^4$. Then $p(0) = 0$ gives $b_0 = 0$, and the remaining two conditions are

$$\begin{aligned} p(1/2) &= \tfrac{1}{4} b_2 + \tfrac{1}{16} b_4 = \tfrac{1}{2}, \\ p(1) &= b_2 + b_4 = 1. \end{aligned}$$

Multiplying the first by 16 gives $4b_2 + b_4 = 8$; subtracting the second yields $3b_2 = 7$, so $b_2 = 7/3$ and $b_4 = -4/3$. (Check: $p(\pm 1/2) = 7/12 - 1/12 = 1/2$.)
Sketch: this quartic meets $|x|$ at all five nodes and hugs the V much more closely than x^2 did. It still sags below $|x|$ on $(0, 1/2)$ (for instance $p(1/4) = 9/64$ against $|1/4| = 1/4$), but since $p'(x) = \frac{2x}{3}(7 - 8x^2)$ it has interior maxima at $x = \pm\sqrt{7/8} \approx \pm 0.935$, where

$$p\big(\pm\sqrt{7/8}\big) = \frac{49}{48} \approx 1.021 > 0.935 = |x|,$$

so it bulges above the V near the endpoints.

6.9 Exercises 6.7.4–6.7.10

Problem 6.7.4. Show that $f(x) = \sqrt{1 - x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution. Differentiate $(1 - x)^{1/2}$ n times and feed the result into the defining formula $a_n = f^{(n)}(0)/n!$ for Taylor coefficients (p. 200; Theorem 6.6.2 is unavailable here, since a power series representation for f is exactly what is not yet known).
Since $\frac{d}{dx}(1 - x)^\alpha = -\alpha(1 - x)^{\alpha - 1}$, an induction gives, for all $n \geq 0$ and $x < 1$,

$$f^{(n)}(x) = (-1)^n \Big(\prod_{k=0}^{n-1} (\tfrac{1}{2} - k) \Big) (1 - x)^{1/2 - n},$$

the empty product for $n=0$ being 1. Setting $x=0$ leaves

$$f^{(n)}(0) = (-1)^n \prod_{k=0}^{n-1} \left(\tfrac{1}{2} - k\right).$$

For $n \geq 1$ pull a factor $1/2$ out of each of the n terms:

$$\prod_{k=0}^{n-1} \left(\tfrac{1}{2} - k\right) = \frac{1 \cdot (-1) \cdot (-3) \cdots \bigl(-(2n-3)\bigr)}{2^n} = \frac{(-1)^{n-1}\, 1 \cdot 3 \cdots (2n-3)}{2^n},$$

since exactly $n-1$ of the n numerators carry a minus sign. Hence

$$f^{(n)}(0) = (-1)^n \cdot \frac{(-1)^{n-1}\, 1 \cdot 3 \cdots (2n-3)}{2^n} = -\frac{1 \cdot 3 \cdots (2n-3)}{2^n}.$$

Dividing by $n!$ and absorbing $2^n n! = 2 \cdot 4 \cdot 6 \cdots 2n$ into the denominator,

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n} \qquad (n \geq 1),$$

while $a_0 = f(0) = 1$. (For $n=1$ the product $1 \cdot 3 \cdots (2n-3)$ is empty, hence 1, and $a_1 = -1/2$; for $n=2$, $a_2 = -1/8$.)

Problem 6.7.5. (a) Follow the advice in Exercise 6.6.9 to prove the Cauchy form of the remainder:

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.
(b) Use this result to prove equation (1) is valid for all $x \in (-1,1)$.
(Here equation (1) is $\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$ with the coefficients a_n of Exercise 6.7.4, and $E_N(x) = \sqrt{1-x} - \sum_{n=0}^{N} a_n x^n$. Exercise 6.6.9 advises: for f differentiable $N+1$ times on $(-R,R)$ and $a \in (-R,R)$, set $S_N(x,a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and $E_N(x,a) = f(x) - S_N(x,a)$; fix $x \neq 0$, compute $E_N(x,x)$, differentiate $E_N(x,a)$ with respect to a, and apply the Mean Value Theorem.)

Solution. (a) Apply the Mean Value Theorem to $a \mapsto E_N(x,a)$ on the interval between 0 and x. Fix $x \neq 0$ in $(-R,R)$. First, every $n \geq 1$ term of $S_N(x,a)$ carries a factor $(x-a)^n$, so

$$E_N(x,x) = f(x) - S_N(x,x) = f(x) - f(x) = 0.$$

Second, $a \mapsto E_N(x,a)$ is differentiable on $(-R,R)$ because each $f^{(n)}$, $n \leq N$, is differentiable there (f is $N+1$ times differentiable). Differentiating termwise in a, the $n=0$ term contributes $f'(a)$ and each $n \geq 1$ term contributes

$$\frac{f^{(n+1)}(a)}{n!}(x-a)^n - \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1},$$

so the sum telescopes and only the last positive piece survives:

$$\frac{\partial}{\partial a} S_N(x,a) = \frac{f^{(N+1)}(a)}{N!}(x-a)^N, \qquad E_N'(x,a) = -\frac{f^{(N+1)}(a)}{N!}(x-a)^N.$$

Now $a \mapsto E_N(x,a)$ is continuous on the closed interval with endpoints 0 and x and differentiable on its interior, so the Mean Value Theorem (Theorem 5.3.2) supplies a c strictly between 0 and x with

$$E_N(x,x) - E_N(x,0) = E_N'(x,c)\,(x-0).$$

Since $E_N(x,x) = 0$ and $E_N(x,0) = E_N(x)$, this reads

$$-E_N(x) = -\frac{f^{(N+1)}(c)}{N!}(x-c)^N x, \qquad \text{i.e.} \qquad E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x.$$

(For $x = 0$ the formula is trivially true, both sides being 0.)
(b) Part (a) applies to $f(x) = \sqrt{1-x}$ with $R = 1$, since f is infinitely differentiable on $(-1, 1)$ (every $(1-x)^{1/2-n}$ is defined there). Its derivative formula from Exercise 6.7.4 gives, for $c < 1$,

$$\left|f^{(N+1)}(c)\right| = \frac{1 \cdot 3 \cdots (2N-1)}{2^{N+1}} (1-c)^{-N-1/2},$$

because $\prod_{k=0}^{N} \left|\frac{1}{2} - k\right| = 2^{-(N+1)}\, 1 \cdot 3 \cdots (2N-1)$. Substituting into (a) and writing $2^{N+1}N! = 2\,(2 \cdot 4 \cdots 2N)$,

$$\left|E_N(x)\right| = \frac{1 \cdot 3 \cdots (2N-1)}{2 \cdot 4 \cdots 2N} \cdot \frac{|x|}{2} \cdot \frac{1}{\sqrt{1-c}} \cdot \left|\frac{x-c}{1-c}\right|^N,$$

in which the first factor is a ratio of N matched terms and so is at most 1. The whole point is that the last factor is controlled: for c between 0 and x with $|x| < 1$,

$$\left|\frac{x-c}{1-c}\right| \leq |x|.$$

Indeed, (i) if $0 < c < x < 1$ then $x - c \geq 0$ and $x - c \leq x - xc = x(1-c)$ since $xc \leq c$; (ii) if $-1 < x < c < 0$ then $|x - c| = c - x$ and $c - x \leq -x + xc = -x(1-c) = |x|(1-c)$ since multiplying $x \leq 1$ by the negative number c reverses it to $xc \geq c$. Finally $1 - c$ lies between $\min\{1, 1-x\}$ and $\max\{1, 1-x\}$, so $(1-c)^{-1/2} \leq M_x := \max\{1, (1-x)^{-1/2}\}$, a constant depending only on x. Therefore

$$\left|E_N(x)\right| \;\leq\; \frac{M_x}{2}\, |x|^{N+1} \;\longrightarrow\; 0$$

as $N \to \infty$, since $|x| < 1$ (Example 2.5.3). Hence the partial sums $\sum_{n=0}^{N} a_n x^n$ converge to $\sqrt{1-x}$, which is equation (1), for every $x \in (-1, 1)$.

Problem 6.7.6. (a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$. Show $c_n < \dfrac{2}{\sqrt{2n+1}}$.
(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Exercise 6.7.4.
(c) Carefully explain how this verifies that equation (1) holds for all $x \in [-1, 1]$.

Solution. (a) In fact $c_n < 1/\sqrt{2n+1}$, which is half the stated bound. Square and telescope:

$$c_n^2 = \prod_{k=1}^{n} \frac{(2k-1)^2}{(2k)^2} < \prod_{k=1}^{n} \frac{2k-1}{2k+1} = \frac{1}{2n+1},$$

where the middle inequality is termwise, being equivalent to $(2k-1)(2k+1) < (2k)^2$, i.e. $4k^2 - 1 < 4k^2$. Taking square roots,

$$c_n < \frac{1}{\sqrt{2n+1}} < \frac{2}{\sqrt{2n+1}}.$$

(b) The two products differ by a single factor: for $n \geq 1$,

$$|a_n| = \frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots 2n} = \frac{c_n}{2n-1},$$

(the $n = 1$ case reading $|a_1| = c_1/1 = 1/2$). Hence, using (a) together with $2n - 1 \geq n$ and $\sqrt{2n+1} > \sqrt{2n}$,

$$|a_n| < \frac{2}{(2n-1)\sqrt{2n+1}} \leq \frac{2}{n \cdot \sqrt{2n}} = \frac{\sqrt{2}}{n^{3/2}}.$$

Since $\sum 1/n^{3/2}$ converges (p-series with $p = 3/2 > 1$, Corollary 2.4.7), the Comparison Test (Theorem 2.7.4) gives $\sum_{n=0}^{\infty} |a_n| < \infty$; absolute convergence implies convergence of $\sum a_n$ (Theorem 2.7.6).
(c) Set $M_n = |a_n|$. For every $x \in [-1, 1]$ we have $|a_n x^n| \le |a_n| = M_n$, and $\sum M_n$ converges by (b), so the Weierstrass M-Test (Corollary 6.4.5) says

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges uniformly on $[-1, 1]$. Each partial sum is a polynomial, hence continuous, so g is continuous on all of $[-1, 1]$ by the Term-by-term Continuity Theorem (Theorem 6.4.2). The function $\sqrt{1-x}$ is also continuous on $[-1, 1]$, and by Exercise 6.7.5(b) the two agree on $(-1, 1)$.
Two continuous functions agreeing on $(-1, 1)$ agree on the closure: taking $x_k \to 1^-$ with $x_k \in (-1, 1)$, sequential continuity (Theorem 4.3.2) gives

$$g(1) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} \sqrt{1 - x_k} = 0 = \sqrt{1-1},$$

and the same argument at $x = -1$ with $x_k \to -1^+$ gives $g(-1) = \sqrt{2}$. So equation (1) holds at both endpoints as well, hence for all $x \in [-1, 1]$.

Problem 6.7.7. (a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\epsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$\big|\, |x| - q(x) \,\big| < \epsilon$$

for all $x \in [-1, 1]$.
(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

Solution. (a) Substitute $1 - x^2$ for x in equation (1) and truncate:

$$q(x) = \sum_{n=0}^{N} a_n \,(1 - x^2)^n$$

for N large enough, with a_n the coefficients of Exercise 6.7.4.
This is legitimate because $x \in [-1, 1]$ forces $x^2 \in [0, 1]$ and therefore $1 - x^2 \in [0, 1] \subseteq [-1, 1]$, exactly the range on which Exercise 6.7.6(c) established equation (1). Evaluating (1) at the point $1 - x^2$,

$$|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)} = \sum_{n=0}^{\infty} a_n (1 - x^2)^n.$$

Each $q(x)$ above is a polynomial in x (of degree $2N$). Since $0 \le 1 - x^2 \le 1$, the tail estimate is uniform in x:

$$\big|\, |x| - q(x) \,\big| = \left| \sum_{n=N+1}^{\infty} a_n (1 - x^2)^n \right| \le \sum_{n=N+1}^{\infty} |a_n|.$$

By Exercise 6.7.6(b) the series $\sum |a_n|$ converges, so its tails tend to 0; choose N with $\sum_{n>N} |a_n| < \epsilon$ and the resulting q works for every $x \in [-1, 1]$ simultaneously.
(b) Rescale: let $M = \max\{|a|, |b|\}$, which is positive since $a < b$, and put

$$p(x) = M\, q(x/M),$$

where q is chosen by part (a) for the tolerance ϵ/M on $[-1, 1]$.
Then p is a polynomial, and $x \in [a, b]$ implies $|x| \le M$, i.e. $x/M \in [-1, 1]$, so

$$\big|\, |x| - p(x) \,\big| = M \left| \left|\frac{x}{M}\right| - q\Big(\frac{x}{M}\Big) \right| < M \cdot \frac{\epsilon}{M} = \epsilon$$

for all $x \in [a, b]$.

Problem 6.7.8. (a) Fix $a \in [-1,1]$ and sketch

$$h_a(x) = \frac{1}{2}\big(|x-a| + (x-a)\big)$$

over $[-1,1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1,a]$.
(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1,1]$.
(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1,1]$.
(d) Complete the proof of WAT for the interval $[-1,1]$, and then generalize to an arbitrary interval $[a,b]$.

Solution. (a) h_a is the ramp that switches on at a:

$$h_a(x) = \begin{cases} \frac{1}{2}\big((a-x) + (x-a)\big) = 0, & -1 \le x \le a, \\ \frac{1}{2}\big((x-a) + (x-a)\big) = x-a, & a \le x \le 1. \end{cases}$$

Its graph is the segment along the axis from $(-1,0)$ to $(a,0)$ followed by the segment of slope 1 from $(a,0)$ to $(1,1-a)$, so h_a is polygonal for the partition $-1 < a < 1$ (and is a single line segment when $a = \pm 1$).
(b) Because h_a differs from a multiple of an absolute value by a linear function. Let $\epsilon > 0$. For $x \in [-1,1]$ and $a \in [-1,1]$ we have $x - a \in [-2,2]$, so by Exercise 6.7.7(b) applied to the interval $[-2,2]$ there is a polynomial q with

$$\big|\,|t| - q(t)\,\big| < 2\epsilon \quad \text{for all } t \in [-2,2].$$

Put $p(x) = \frac{1}{2}\big(q(x-a) + (x-a)\big)$, a polynomial in x. Then for every $x \in [-1,1]$,

$$|h_a(x) - p(x)| = \tfrac{1}{2}\big|\,|x-a| - q(x-a)\,\big| < \epsilon.$$

(c) Take $b_0 = m_1$ and $b_i = m_{i+1} - m_i$ for $1 \le i \le n-1$, where m_i is the slope of ϕ on $[a_{i-1}, a_i]$. Set

$$\psi(x) = \phi(-1) + \sum_{i=0}^{n-1} b_i h_{a_i}(x).$$

Fix k with $1 \le k \le n$ and let $x \in [a_{k-1}, a_k]$. By (a), $h_{a_i}(x) = x - a_i$ for $i \le k-1$ and $h_{a_i}(x) = 0$ for $i \ge k$, so ψ is linear on $[a_{k-1}, a_k]$ with slope

$$\sum_{i=0}^{k-1} b_i = m_1 + \sum_{i=1}^{k-1} (m_{i+1} - m_i) = m_k,$$

the sum telescoping. Thus ψ is polygonal for the same partition and has the same slope as ϕ on each subinterval. Also $h_{a_i}(-1) = 0$ for every i (since $a_i \ge -1$), so $\psi(-1) = \phi(-1)$. Now induct on k: if $\psi(a_{k-1}) = \phi(a_{k-1})$ then on $[a_{k-1}, a_k]$ both functions equal $\phi(a_{k-1}) + m_k(x - a_{k-1})$, which gives $\psi = \phi$ there and in particular $\psi(a_k) = \phi(a_k)$. Hence $\psi = \phi$ on all of $[-1,1]$.
(d) Let $f : [-1,1] \to \mathbf{R}$ be continuous and let $\epsilon > 0$. By Theorem 6.7.3 there is a polygonal ϕ with $|f(x) - \phi(x)| < \epsilon/2$ on $[-1,1]$; its partition runs from -1 to 1 (Definition 6.7.2), so (c) writes it with

constants $b_0, \dots, b_{n-1}$. Set $M = 1 + \sum_{i=0}^{n-1} |b_i|$. By (b) choose polynomials p_i with $|h_{a_i}(x) - p_i(x)| < \epsilon/(2M)$ on $[-1,1]$, and put $p(x) = \phi(-1) + \sum_{i=0}^{n-1} b_i p_i(x)$, a polynomial. Then

$$\begin{aligned}|\phi(x) - p(x)| &\le \sum_{i=0}^{n-1} |b_i| \left|h_{a_i}(x) - p_i(x)\right| \\ &< \frac{\epsilon}{2M} \sum_{i=0}^{n-1} |b_i| < \frac{\epsilon}{2},\end{aligned}$$

and the triangle inequality gives $|f(x) - p(x)| < \epsilon$ for all $x \in [-1,1]$.
For a general interval $[a,b]$, let $L(t) = a + \frac{b-a}{2}(t+1)$, the increasing affine bijection of $[-1,1]$ onto $[a,b]$, with affine inverse $L^{-1}(x) = \frac{2(x-a)}{b-a} - 1$. Given f continuous on $[a,b]$, the composition $f \circ L$ is continuous on $[-1,1]$, so the case just proved supplies a polynomial p with $|f(L(t)) - p(t)| < \epsilon$ for all $t \in [-1,1]$. Then $P = p \circ L^{-1}$ is again a polynomial, and substituting $t = L^{-1}(x)$ gives $|f(x) - P(x)| < \epsilon$ for all $x \in [a,b]$.

Problem 6.7.9. (a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a,b]$ with the open interval (a,b).
(b) What happens if we replace $[a,b]$ with the closed set $[a,\infty)$. Does the theorem still hold?

Solution. (a) $f(x) = 1/x$ on $(0,1)$. It is continuous there, but unbounded, while every polynomial p is continuous on the compact set $[0,1]$ and hence bounded there (Theorem 4.4.2), say $|p| \le K$ on $[0,1]$. Taking $\epsilon = 1$, at any $x \in (0, 1/(K+1))$ we get

$$|f(x) - p(x)| \ge \frac{1}{x} - K > (K+1) - K = 1,$$

so no polynomial approximates f to within 1 on $(0,1)$.
(b) No. Take $f(x) = \sin x$ on $[0,\infty)$, continuous and bounded, and let $\epsilon = 1/2$. If $|f(x) - p(x)| < 1/2$ for all $x \ge 0$, then $|p(x)| < 3/2$ on $[0,\infty)$; a polynomial of degree ≥ 1 satisfies $|p(x)| \to \infty$ as $x \to \infty$, so p must be a constant c. But sin attains both 1 (at $x = \pi/2$) and -1 (at $x = 3\pi/2$), whence

$$2 = |1 - (-1)| \le |1 - c| + |c - (-1)| < \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

a contradiction.

Problem 6.7.10. Is there a countable subset of polynomials $\mathcal{C}$ with the property that every continuous function on $[a,b]$ can be uniformly approximated by polynomials from $\mathcal{C}$?

Solution. Yes: let $\mathcal{C}$ be the set of polynomials with rational coefficients.
$\mathcal{C}$ is countable. Let $\mathcal{C}_n \subseteq \mathcal{C}$ consist of those of degree at most n. Then $\mathcal{C}_0$ is in one-to-one correspondence with $\mathbf{Q}$, which is countable by Theorem 1.5.6(i), and

$$\mathcal{C}_n = \bigcup_{r \in \mathbf{Q}} \{\, p + r x^n : p \in \mathcal{C}_{n-1} \,\}$$

exhibits $\mathcal{C}_n$ as a countable union of countable sets, hence countable by Theorem 1.5.8(ii); induction gives every $\mathcal{C}_n$ countable, and $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$ is countable by Theorem 1.5.8(ii) once more.
$\mathcal{C}$ suffices. Let f be continuous on $[a,b]$ and let $\epsilon > 0$. By WAT (Theorem 6.7.1) there is a polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

with $|f(x) - p(x)| < \epsilon/2$ on $[a,b]$. Set $M = \max\{1, |a|, |b|\}$, so that $|x|^k \le M^n$ for $0 \le k \le n$ and $x \in [a,b]$. By density of $\mathbf{Q}$ (Theorem 1.4.3) choose $r_k \in \mathbf{Q}$ with

$$|c_k - r_k| < \frac{\epsilon}{2(n+1)M^n}, \qquad 0 \le k \le n,$$

and put $q(x) = r_0 + r_1 x + \cdots + r_n x^n \in \mathcal{C}$. Then for all $x \in [a, b]$,

$$
\begin{aligned}
|p(x) - q(x)| &\le \sum_{k=0}^{n} |c_k - r_k| \, |x|^k \\
&\le M^n \sum_{k=0}^{n} |c_k - r_k| < \frac{\epsilon}{2},
\end{aligned}
$$

so $|f(x) - q(x)| < \epsilon$ on $[a, b]$.

6.10 Exercises 6.7.11–6.7.11

Problem 6.7.11. Assume that f has a continuous derivative on $[a, b]$. Show that, given $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \epsilon \quad \text{and} \quad |f'(x) - p'(x)| < \epsilon$$

for all $x \in [a, b]$.

Solution. Approximate f' and then antidifferentiate. Set $\epsilon' = \epsilon/(1 + b - a)$, so that $\epsilon' \le \epsilon$ and $\epsilon'(b - a) < \epsilon$. Since f' is continuous on $[a, b]$, WAT (Theorem 6.7.1) provides a polynomial

$$q(x) = c_0 + c_1 x + \cdots + c_n x^n \quad \text{with} \quad |f'(x) - q(x)| < \epsilon'$$

for all $x \in [a, b]$. Define the polynomial

$$p(x) = f(a) + \sum_{k=0}^{n} \frac{c_k}{k+1} \big(x^{k+1} - a^{k+1}\big),$$

so that $p(a) = f(a)$ and $p'(x) = q(x)$ for every x by the algebraic differentiation rules of Theorem 5.2.4. The derivative estimate is immediate: $|f'(x) - p'(x)| = |f'(x) - q(x)| < \epsilon' \le \epsilon$ on $[a, b]$.
For the other estimate put $g = f - p$. Then $g(a) = 0$, and g is continuous on $[a, b]$ and differentiable on (a, b) (indeed on all of $[a, b]$), so the Mean Value Theorem (Theorem 5.3.2) applies on $[a, x]$ for each $x \in (a, b]$ and yields $c \in (a, x)$ with

$$
\begin{aligned}
|f(x) - p(x)| &= |g(x) - g(a)| = |g'(c)| \, (x - a) \\
&= |f'(c) - q(c)| \, (x - a) < \epsilon'(b - a) < \epsilon.
\end{aligned}
$$

At $x = a$ the difference is 0, so $|f(x) - p(x)| < \epsilon$ holds for all $x \in [a, b]$.

7 The Riemann Integral

Contents

7.1 Exercises 7.2.1–7.2.7

Problem 7.2.1. Let f be a bounded function on $[a,b]$, and let P be an arbitrary partition of $[a,b]$. First, explain why $U(f) \geq L(f,P)$. Now, prove Lemma 7.2.6.

Solution. $L(f,P)$ is a lower bound for the entire set of upper sums: by Lemma 7.2.4, $L(f,P) \leq U(f,Q)$ for every partition Q. Hence $L(f,P)$ is at most the greatest lower bound, which is $U(f)$ by Definition 7.2.5:

$$L(f,P) \leq \inf\{U(f,Q) : Q \in \mathcal{P}\} = U(f),$$

where $\mathcal{P}$ is the collection of all partitions of $[a,b]$ (that set of upper sums is nonempty and bounded below by $L(f,P)$, so the infimum exists by the Axiom of Completeness).
(Lemma 7.2.6.) The displayed inequality holds for every $P \in \mathcal{P}$, so $U(f)$ is an upper bound for the nonempty set $\{L(f,P) : P \in \mathcal{P}\}$, and is therefore at least its least upper bound:

$$U(f) \geq \sup\{L(f,P) : P \in \mathcal{P}\} = L(f). \qquad \blacksquare$$

Problem 7.2.2. Consider $f(x) = 1/x$ over the interval $[1,4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.
(a) Compute $L(f,P)$, $U(f,P)$, and $U(f,P) - L(f,P)$.
(b) What happens to the value of $U(f,P) - L(f,P)$ when we add the point 3 to the partition?
(c) Find a partition P' of $[1,4]$ for which $U(f,P') - L(f,P') < 2/5$.

Solution. $L(f,P) = 13/12$, $U(f,P) = 11/6$, and the difference is $3/4$.
Since $f(x) = 1/x$ is decreasing, on each $[x_{k-1}, x_k]$ we have $m_k = 1/x_k$ and $M_k = 1/x_{k-1}$. With $\Delta x = (1/2, 1/2, 2)$:

$$\begin{aligned} L(f,P) &= \tfrac{2}{3}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{4}\cdot 2 = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{13}{12}, \\ U(f,P) &= 1\cdot\tfrac{1}{2} + \tfrac{2}{3}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot 2 = \tfrac{1}{2} + \tfrac{1}{3} + 1 = \tfrac{11}{6}, \\ U(f,P) - L(f,P) &= \tfrac{22}{12} - \tfrac{13}{12} = \tfrac{3}{4}. \end{aligned}$$

(b) It drops to $1/2$. Only the last subinterval changes, and its contribution to $U - L$ falls from $\left(\tfrac{1}{2} - \tfrac{1}{4}\right)\cdot 2 = \tfrac{1}{2}$ to

$$\left(\tfrac{1}{2} - \tfrac{1}{3}\right)\cdot 1 + \left(\tfrac{1}{3} - \tfrac{1}{4}\right)\cdot 1 = \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{1}{4},$$

so $U - L = \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2}$.
(c) Take $P' = \{1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4\}$, the uniform partition into $n = 6$ pieces of width $\Delta x = 1/2$. Because f is decreasing, $M_k - m_k = f(x_{k-1}) - f(x_k)$ and the sum telescopes:

$$U(f,P') - L(f,P') = \tfrac{1}{2}\sum_{k=1}^{6}\big(f(x_{k-1}) - f(x_k)\big) = \tfrac{1}{2}\left(1 - \tfrac{1}{4}\right) = \tfrac{3}{8} < \tfrac{2}{5}.$$

Problem 7.2.3. (Sequential Criterion for Integrability).
(a) Prove that a bounded function f is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty}\big[U(f,P_n) - L(f,P_n)\big] = 0,$$

and in this case $\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$.
(b) For each n, let P_n be the partition of $[0,1]$ into n equal subintervals. Find formulas for $U(f,P_n)$ and $L(f,P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n+1)/2$ will be useful.
(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

Solution. (a) ($\Rightarrow$) If f is integrable, apply Theorem 7.2.8 with $\epsilon = 1/n$ to get a partition P_n with $U(f,P_n) - L(f,P_n) < 1/n$; these differences are nonnegative, so they tend to 0 by the Squeeze Theorem (Theorem 2.3.3).
($\Leftarrow$) Given such a sequence, note $L(f,P_n) \le L(f) \le U(f) \le U(f,P_n)$ by Definition 7.2.5 and Lemma 7.2.6. Hence for every n,

$$0 \le U(f) - L(f) \le U(f,P_n) - L(f,P_n),$$

and the right side tends to 0, forcing $U(f) = L(f)$: f is integrable (Definition 7.2.7). The same sandwich, now with $\int_a^b f = U(f) = L(f)$ in the middle, gives

$$0 \le U(f,P_n) - \int_a^b f \le U(f,P_n) - L(f,P_n) \longrightarrow 0,$$

so $U(f,P_n) \to \int_a^b f$, and likewise $L(f,P_n) \to \int_a^b f$.
(b) Here $x_k = k/n$ and $\Delta x_k = 1/n$; $f(x) = x$ is increasing, so $M_k = k/n$ and $m_k = (k-1)/n$. Therefore

$$U(f,P_n) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n},$$
$$L(f,P_n) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n}.$$

(c) $U(f,P_n) - L(f,P_n) = \dfrac{n+1}{2n} - \dfrac{n-1}{2n} = \dfrac{1}{n} \to 0$, so by (a) f is integrable and

$$\int_0^1 f = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Problem 7.2.4. Let g be bounded on $[a,b]$ and assume there exists a partition P with $L(g,P) = U(g,P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Solution. g is constant on $[a,b]$; it is integrable with $\int_a^b g = g(a)(b-a)$.
Write $P = \{a = x_0, x_1, \dots, x_n = b\}$. Since

$$U(g,P) - L(g,P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = 0$$

is a sum of nonnegative terms with $\Delta x_k > 0$, every $M_k = m_k$. So g takes a single value c_k on the **closed** subinterval $[x_{k-1}, x_k]$. Consecutive subintervals share the endpoint x_k, whence

$$c_k = g(x_k) = c_{k+1} \qquad (1 \le k \le n-1),$$

so all the c_k coincide and $g \equiv c = g(a)$ on $[a,b]$.
Integrability is then immediate from Theorem 7.2.8 (the hypothesis supplies a partition with $U - L = 0 < \epsilon$ for every $\epsilon > 0$), and

$$\int_a^b g = U(g,P) = \sum_{k=1}^{n} c\,\Delta x_k = c(b-a) = g(a)(b-a).$$

Problem 7.2.5. Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n) \to f$ uniformly on $[a,b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Solution. Uniform closeness transfers the Integrability Criterion: a partition that works for f_N works for f, up to $2\eta(b-a)$.
Let $\epsilon > 0$ and set $\eta = \dfrac{\epsilon}{4(b-a)}$. Choose N with

$$|f_N(x) - f(x)| < \eta \qquad \text{for all } x \in [a,b].$$

(In particular f is bounded, since f_N is.) Then $f_N - \eta < f < f_N + \eta$ pointwise, so on any subinterval $[x_{k-1}, x_k]$ of any partition P the suprema and infima satisfy

$$M_k(f) \leq M_k(f_N) + \eta, \qquad m_k(f) \geq m_k(f_N) - \eta,$$

and therefore $M_k(f) - m_k(f) \leq \big(M_k(f_N) - m_k(f_N)\big) + 2\eta$. Multiplying by Δx_k and summing,

$$U(f,P) - L(f,P) \leq \big[U(f_N,P) - L(f_N,P)\big] + 2\eta(b-a).$$

Now use the hypothesis that f_N is integrable: by the Integrability Criterion (Theorem 7.2.8) there is a partition P_ϵ with $U(f_N, P_\epsilon) - L(f_N, P_\epsilon) < \epsilon/2$. For this partition,

$$\begin{aligned} U(f,P_\epsilon) - L(f,P_\epsilon) &< \frac{\epsilon}{2} + 2\eta(b-a) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, Theorem 7.2.8 (applied now to the bounded function f) gives that f is integrable on $[a,b]$.

Problem 7.2.6. A tagged partition $(P, \{c_k\})$ is one where in addition to a partition P we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. The corresponding Riemann sum,

$$R(f,P) = \sum_{k=1}^{n} f(c_k)\,\Delta x_k,$$

is discussed in Section 7.1, where the following definition is alluded to.
Riemann's Original Definition of the Integral: A bounded function f is integrable on $[a,b]$ with $\int_a^b f = A$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f,P) - A| < \epsilon.$$

Show that if f satisfies Riemann's definition above, then f is integrable in the sense of Definition 7.2.7. (The full equivalence of these two characterizations of integrability is proved in Section 8.1.)

Solution. Sample the tags at near-suprema and at near-infima of the same partition; the two Riemann sums pin $U(f,P)$ and $L(f,P)$ to within ϵ of each other.
Let $\epsilon > 0$. Riemann's definition, applied with $\epsilon/4$, supplies $\delta > 0$. Fix any partition $P = \{x_0, \ldots, x_n\}$ with $\Delta x_k < \delta$ for all k (a uniform partition with $n > (b-a)/\delta$ will do), and put $\eta = \dfrac{\epsilon}{4(b-a)}$.
(i) Since $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$, the number $M_k - \eta$ is not an upper bound, so we may choose a tag $c_k \in [x_{k-1}, x_k]$ with $f(c_k) > M_k - \eta$. The tagged partition $(P, \{c_k\})$ satisfies $\Delta x_k < \delta$, so

$$\begin{aligned} U(f,P) - \eta(b-a) &< \sum_{k=1}^{n} f(c_k)\Delta x_k = R(f,P) \\ &< A + \frac{\epsilon}{4}. \end{aligned}$$

Since $\eta(b-a) = \epsilon/4$, this gives $U(f,P) < A + \epsilon/2$.

(ii) Symmetrically, choose tags $d_k \in [x_{k-1}, x_k]$ with $f(d_k) < m_k + \eta$. The same δ applies, so

$$\begin{aligned} L(f,P) + \eta(b-a) &> \sum_{k=1}^{n} f(d_k)\Delta x_k \\ &> A - \frac{\epsilon}{4}, \end{aligned}$$

so $L(f,P) > A - \epsilon/2$.
Subtracting, $U(f,P) - L(f,P) < \epsilon$. As $\epsilon > 0$ was arbitrary and f is bounded by hypothesis, the Integrability Criterion (Theorem 7.2.8) shows f is integrable in the sense of Definition 7.2.7.

Problem 7.2.7. Let $f : [a,b] \to \mathbf{R}$ be increasing on the set $[a,b]$ (i.e., $f(x) \le f(y)$ whenever $x < y$). Show that f is integrable on $[a,b]$.

Solution. Monotonicity makes the upper minus lower sum telescope on a uniform partition.
First, f is bounded: $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$.
Given $\epsilon > 0$, choose $n \in \mathbf{N}$ with

$$n > \frac{(b-a)\big(f(b) - f(a)\big)}{\epsilon},$$

and let $P_n = \{x_0, \dots, x_n\}$ be the uniform partition, $x_k = a + k(b-a)/n$, so $\Delta x_k = (b-a)/n$. Because f is increasing, the supremum and infimum on $[x_{k-1}, x_k]$ are attained at the endpoints: $M_k = f(x_k)$ and $m_k = f(x_{k-1})$. Hence

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)\frac{b-a}{n} \\ &= \frac{b-a}{n}\big(f(b) - f(a)\big) < \epsilon, \end{aligned}$$

the middle equality by telescoping. Since $\epsilon > 0$ was arbitrary, f is integrable by the Integrability Criterion (Theorem 7.2.8).

7.2 Exercises 7.3.1–7.3.7

Problem 7.3.1. Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0,1]$.
(a) Show that $L(f,P) = 1$ for every partition P of $[0,1]$.
(b) Construct a partition P for which $U(f,P) < 1 + 1/10$.
(c) Given $\epsilon > 0$, construct a partition P_ϵ for which $U(f,P_\epsilon) < 1 + \epsilon$.

Solution. (The parts write f for the function displayed as h; they are the same function.)
(a) Every m_k equals 1. Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$. Each subinterval $[x_{k-1}, x_k]$ has positive length, hence meets $[0,1)$, where $h = 1$; since $h \ge 1$ on all of $[0,1]$,

$$m_k = \inf\{h(x) : x \in [x_{k-1}, x_k]\} = 1,$$

and therefore

$$L(h,P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1.$$

(b) Take $P = \{0, 19/20, 1\}$. Then $M_1 = 1$ and $M_2 = 2$, so

$$U(h,P) = 1 \cdot \tfrac{19}{20} + 2 \cdot \tfrac{1}{20} = \tfrac{21}{20} = 1 + \tfrac{1}{20} < 1 + \tfrac{1}{10}.$$

(c) Take $P_\epsilon = \{0, 1-\delta, 1\}$ with $\delta = \frac{1}{2}\min\{1, \epsilon\}$. The supremum of h is 1 on $[0, 1-\delta]$ and 2 on $[1-\delta, 1]$, so

$$U(h, P_\epsilon) = 1 \cdot (1-\delta) + 2 \cdot \delta = 1 + \delta < 1 + \epsilon.$$

Problem 7.3.2. Recall that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0, 1]$ with $\int_0^1 t = 0$.
(a) First argue that $L(t, P) = 0$ for any partition P of $[0, 1]$.
(b) Let $\epsilon > 0$, and consider the set of points $D_{\epsilon/2} = \{x \in [0, 1] : t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?
(c) To complete the argument, explain how to construct a partition P_ϵ of $[0, 1]$ so that $U(t, P_\epsilon) < \epsilon$.

Solution. (a) Every m_k is 0. Each subinterval $[x_{k-1}, x_k]$ of a partition has positive length and so contains an irrational point, where $t = 0$; since $t \geq 0$ everywhere, $m_k = 0$ for all k and

$$L(t, P) = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0.$$

Consequently $L(t) = \sup_P L(t, P) = 0$.
(b) It is finite. If $t(x) \geq \epsilon/2 > 0$ then either $x = 0$, or $x = m/n$ in lowest terms with $1/n \geq \epsilon/2$, i.e. $n \leq 2/\epsilon$. For each such n there are at most $n + 1$ points of the form m/n in $[0, 1]$, so

$$\#D_{\epsilon/2} \ \leq\ 1 + \sum_{n \leq 2/\epsilon} (n+1) \ <\ \infty.$$

Write $N = \#D_{\epsilon/2}$.
(c) Let P_ϵ be the partition of $[0, 1]$ into n equal subintervals, where $n > 8(N+1)/\epsilon$. Split the subintervals into those that meet $D_{\epsilon/2}$ and those that do not. Each of the N points of $D_{\epsilon/2}$ lies in at most two subintervals, so at most $2N$ subintervals are of the first kind; on each of them $M_k \leq 1$ because $0 \leq t \leq 1$. On a subinterval of the second kind every value of t is $< \epsilon/2$, so $M_k \leq \epsilon/2$. Hence

$$\begin{aligned} U(t, P_\epsilon) &\leq 1 \cdot 2N \cdot \tfrac{1}{n} \ +\ \tfrac{\epsilon}{2} \sum_k \Delta x_k \\ &< \tfrac{\epsilon}{4} + \tfrac{\epsilon}{2} \ <\ \epsilon. \end{aligned}$$

With (a) this gives $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$ for every $\epsilon > 0$, so t is integrable by Theorem 7.2.8. Moreover $U(t) \leq U(t, P_\epsilon) < \epsilon$ for every ϵ, so $U(t) = 0$, and

$$\int_0^1 t = U(t) = L(t) = 0.$$

Problem 7.3.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. $\int_0^1 f = 0$.
First, $L(f, P) = 0$ for every partition P of $[0, 1]$: each subinterval has positive length and so contains

an irrational point, where $f = 0$, while $f \geq 0$ throughout; hence every $m_k = 0$ and

$$L(f) = \sup_P L(f, P) = 0.$$

Next fix $c \in (0,1)$. On $[c,1]$ the only points where f is nonzero are those $1/n$ with $1/n \geq c$, i.e. $n \leq 1/c$, and there are at most $\lfloor 1/c \rfloor$ of them. So on $[c,1]$ the function f agrees with the (integrable) zero function at all but finitely many points, and Exercise 7.3.7(a) makes f integrable on $[c,1]$.
Since f is bounded on $[0,1]$ (indeed $0 \leq f \leq 1$) and integrable on $[c,1]$ for every $c \in (0,1)$, Theorem 7.3.2 gives that f is integrable on $[0,1]$. Its value is forced by the lower integral:

$$\int_0^1 f = L(f) = 0.$$

Problem 7.3.4. Let f and g be functions defined on (possibly different) closed intervals, and assume the range of f is contained in the domain of g so that the composition $g \circ f$ is properly defined.
(a) Show, by example, that it is not the case that if f and g are integrable, then $g \circ f$ is integrable.
Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.
(b) If f is increasing and g is integrable, then $g \circ f$ is integrable.
(c) If f is integrable and g is increasing, then $g \circ f$ is integrable.

Solution. (a) Take $f = t$, Thomae's function on $[0,1]$, and $g = \chi_{(0,1]}$ on $[0,1]$:

$$g(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } 0 < y \leq 1. \end{cases}$$

Both are integrable: t by Exercise 7.3.2, and g by Theorem 7.3.2, since g is bounded and continuous (hence integrable, Theorem 7.2.9) on $[c,1]$ for every $c \in (0,1)$. The range of t lies in $[0,1]$, and $t(x) > 0$ exactly when x is rational, so

$$(g \circ t)(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

which is Dirichlet's function, not integrable (Example 7.3.3).
(b) False. Let $\{r_1, r_2, r_3, \ldots\}$ enumerate $\mathbf{Q} \cap [0,1]$ and define

$$f(x) = \sum_{n \,:\, r_n \leq x} 2^{-n}, \qquad x \in [0,1].$$

Then f is strictly increasing (for $x < y$ a rational $r_k \in (x, y]$ gives $f(y) - f(x) \geq 2^{-k} > 0$), with $0 \leq f \leq f(1) = 1$, so its range sits in $[0,1]$.
Write $b_m = f(r_m)$ and $a_m = b_m - 2^{-m}$; the term $n = m$ occurs in b_m, so $0 \leq a_m < b_m \leq 1$. If $x < r_m$ then (x, r_m) contains a rational, whence

$$f(r_m) - f(x) = \sum_{x < r_n \leq r_m} 2^{-n} > 2^{-m},$$

i.e. $f(x) < a_m$; and $x \geq r_m$ gives $f(x) \geq b_m$. Hence

$$f([0,1]) \cap [a_m, b_m) = \emptyset \qquad (m \in \mathbf{N}).$$

In particular no b_k lies in $[a_m, b_m)$, so $r_m < r_k$ gives both $b_m < b_k$ and $b_m \notin [a_k, b_k)$, i.e. $b_m < a_k$: these intervals are pairwise disjoint, of total length $\sum_m 2^{-m} = 1$.
Now put $B = f(\mathbf{Q} \cap [0,1]) = \{b_m : m \in \mathbf{N}\}$ and let $g = \chi_B$ on $[0,1]$. Since B is countable, every subinterval of a partition contains points outside B, so $L(g, P) = 0$ for all P. Given $\epsilon > 0$, choose N with $\sum_{n>N} 2^{-n} < \epsilon/2$, set $\delta = \min\{\epsilon/(4N),\, 2^{-N-1}\}$, and let P consist of 0, 1, and the points

a_m, $b_m - \delta$, b_m for $m \le N$. Each subinterval $[a_m, b_m - \delta]$ lies inside $[a_m, b_m)$, which misses B, and so contributes 0; the remaining subintervals have $M_k \le 1$ and total length

$$\begin{aligned} 1 - \sum_{m \le N} \left(2^{-m} - \delta\right) &= \sum_{m>N} 2^{-m} + N\delta \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{4} < \epsilon. \end{aligned}$$

So $U(g, P) - L(g, P) < \epsilon$ for the bounded function g, and Theorem 7.2.8 applies.
Finally, f is injective, so $f^{-1}(B) = \mathbf{Q} \cap [0, 1]$ and

$$(g \circ f)(x) = \chi_B(f(x)) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

Dirichlet's function again, which is not integrable.
(c) False: in the pair from (a) the function $\chi_{(0,1]}$ is increasing, t is integrable, and their composite is Dirichlet's function.

Problem 7.3.5. Provide an example or give a reason why the request is impossible.
(a) A sequence $(f_n) \to f$ pointwise, where each f_n has at most a finite number of discontinuities but f is not integrable.
(b) A sequence $(g_n) \to g$ uniformly where each g_n has at most a finite number of discontinuities and g is not integrable.
(c) A sequence $(h_n) \to h$ uniformly where each h_n is not integrable but h is integrable.

Solution. (a) Possible. Let $\{r_1, r_2, r_3, \dots\}$ enumerate $\mathbf{Q} \cap [0, 1]$ and set

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Each f_n has exactly n discontinuities. If $x = r_k$ then $f_n(x) = 1$ for all $n \ge k$, and if x is irrational then $f_n(x) = 0$ for all n; so $(f_n) \to f$ pointwise with f Dirichlet's function, which is not integrable (Example 7.3.3).
(b) Impossible. A function bounded on $[a, b]$ with finitely many discontinuities $d_1 < \cdots < d_m$ is integrable: on each closed interval determined by consecutive d_j it is continuous off the two endpoints, hence integrable there by Theorem 7.2.9 together with Theorem 7.3.2 applied at each endpoint, and Theorem 7.4.1 splices the pieces. So every g_n is integrable, and a uniform limit of integrable functions is integrable by Exercise 7.2.5. Hence g is integrable. (Read with the g_n bounded, as Definition 7.2.7 requires of any integrand; the unbounded $g_n(x) = 1/x$ on $(0, 1]$, $g_n(0) = 0$, has one discontinuity and meets the request trivially.)
(c) Possible. Take $h = 0$ on $[0, 1]$ and

$$h_n(x) = \begin{cases} 1/n & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

For every partition P we get $U(h_n, P) = 1/n$ and $L(h_n, P) = 0$, so $U(h_n) = 1/n \ne 0 = L(h_n)$ and h_n is not integrable. Since $|h_n(x) - 0| \le 1/n$ for all x, $(h_n) \to h$ uniformly, and $h = 0$ is integrable.

Problem 7.3.6. Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of all the rationals in $[0, 1]$, and define

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_n \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is $G(x) = \sum_{n=1}^{\infty} g_n(x)$ integrable on $[0, 1]$?

(b) Is $F(x) = \sum_{n=1}^{\infty} g_n(x)/n$ integrable on $[0,1]$?

Solution. (a) No. Each $x \in [0,1]$ equals r_n for at most one index n, so the series defining G has at most one nonzero term and

$$G(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

Dirichlet's function, which is not integrable (Example 7.3.3).
(b) Yes, and $\int_0^1 F = 0$. By the same one-nonzero-term observation,

$$F(x) = \begin{cases} 1/n & \text{if } x = r_n \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

so $0 \le F \le 1$ and the argument of Exercise 7.3.2 applies verbatim. Every subinterval of a partition contains an irrational point, so every $m_k = 0$ and $L(F,P) = 0$ for every P; and for $\epsilon > 0$ the set

$$D_{\epsilon/2} = \{x \in [0,1] : F(x) \ge \epsilon/2\} = \{r_n : n \le 2/\epsilon\}$$

is finite, of cardinality N say. Let P_ϵ split $[0,1]$ into n equal subintervals with $n > 8(N+1)/\epsilon$: at most $2N$ of them meet $D_{\epsilon/2}$, where $M_k \le 1$, while $M_k \le \epsilon/2$ on the others, so

$$\begin{aligned} U(F,P_\epsilon) &\le \frac{2N}{n} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

So $U(F,P_\epsilon) - L(F,P_\epsilon) < \epsilon$, and F is integrable by Theorem 7.2.8; since $U(F) < \epsilon$ for every ϵ,

$$\int_0^1 F = U(F) = L(F) = 0.$$

Problem 7.3.7. Assume $f : [a,b] \to \mathbf{R}$ is integrable.
(a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a,b]$, then g is integrable as well.
(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

Solution. (a) Let $x_1 < x_2 < \cdots < x_m$ be the points where $g \ne f$, with $m \ge 1$ (otherwise $g = f$), and set

$$K = 1 + \sup_{[a,b]} |f| + \max_{1 \le i \le m} |g(x_i)|,$$

finite because f is bounded (Definition 7.2.7), positive, and a bound for both $|f|$ and $|g|$ on $[a,b]$.
Let $\epsilon > 0$. Since f is integrable, Theorem 7.2.8 supplies a partition P_0 with $U(f,P_0) - L(f,P_0) < \epsilon/2$. Put $\lambda = \epsilon/(16Km)$ and let P be the refinement of P_0 obtained by adjoining those of the points $x_i \pm \lambda$ that lie in $[a,b]$. Because $P \supseteq P_0$, Lemma 7.2.3 gives

$$U(f,P) - L(f,P) \le U(f,P_0) - L(f,P_0) < \tfrac{\epsilon}{2}.$$

Call a subinterval of P bad if it contains one of the x_i. Every bad subinterval containing x_i lies inside $[x_i - \lambda, x_i + \lambda]$, so the bad subintervals have total length at most $2m\lambda$. On a subinterval that is not bad, g and f agree and hence have the same supremum and infimum; on a bad one, $M_k(g) - m_k(g) \le 2K$. Therefore

$$\begin{aligned} U(g,P) - L(g,P) &\le \big[U(f,P) - L(f,P)\big] + 2K \cdot 2m\lambda \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} \ < \ \epsilon. \end{aligned}$$

Since ϵ was arbitrary and g is bounded, Theorem 7.2.8 makes g integrable.
(b) Take $f = 0$ on $[0, 1]$ and let g be Dirichlet's function, which equals f except at the countably many rational points of $[0, 1]$. Here f is integrable while g is not (Example 7.3.3).

7.3 Exercises 7.3.8–7.4.5

Problem 7.3.8. As in Exercise 7.3.6, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals in $[0, 1]$, but this time define

$$h_n(x) = \begin{cases} 1 & \text{if } r_n < x \le 1 \\ 0 & \text{if } 0 \le x \le r_n. \end{cases}$$

Show $H(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$ is integrable on $[0, 1]$ even though it has discontinuities at every rational point.

Solution. H is increasing on $[0, 1]$, hence integrable by Exercise 7.2.7. The series converges (indeed $0 \le H \le 1$) by comparison with $\sum 2^{-n}$, and for $x < y$, $x > r_n$ forces $y > r_n$, so $h_n(x) \le h_n(y)$ for every n, and summing against the positive weights 2^{-n} gives $H(x) \le H(y)$.
For the discontinuities, fix a rational $r_n < 1$. Every term of the series $H(x) - H(r_n) = \sum_m \big(h_m(x) - h_m(r_n)\big)/2^m$ is nonnegative by that monotonicity, and the $m = n$ term equals 2^{-n} as soon as $x > r_n$, so

$$H(x) - H(r_n) \ge \frac{1}{2^n} \qquad \text{for all } x \in (r_n, 1],$$

and no right-hand limit at r_n can equal $H(r_n)$. (The printed claim must exclude the rational $x = 1$: there $r_N = 1$ gives $h_N \equiv 0$, and once x exceeds every $r_m < 1$ with $m \le M$ we get $H(1) - H(x) = \sum_{m\,:\,x \le r_m < 1} 2^{-m} \le 2^{-M}$, so H is continuous at 1.)
Method (2): let $H_N = \sum_{n=1}^{N} h_n/2^n$. Each H_N is bounded with finitely many discontinuities, hence integrable by Theorem 7.3.2 together with Theorem 7.4.1 and induction, and

$$\begin{aligned} |H(x) - H_N(x)| &= \sum_{n=N+1}^{\infty} \frac{h_n(x)}{2^n} \\ &\le \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} \end{aligned}$$

for every $x \in [0, 1]$, so $H_N \to H$ uniformly on $[0, 1]$ and Exercise 7.2.5 gives integrability of H.

Problem 7.3.9. (Content Zero). A set $A \subseteq [a, b]$ has *content zero* if for every $\epsilon > 0$ there exists a finite collection of open intervals $\{O_1, O_2, \ldots, O_N\}$ that contain A in their union and whose lengths sum to ϵ or less. Using $|O_n|$ to refer to the length of each interval, we have

$$A \subseteq \bigcup_{n=1}^{N} O_n \qquad \text{and} \qquad \sum_{n=1}^{N} |O_n| \le \epsilon.$$

(a) Let f be bounded on $[a, b]$. Show that if the set of discontinuous points of f has content zero, then f is integrable.
(b) Show that any finite set has content zero.
(c) Content zero sets do not have to be finite. They do not have to be countable. Show that the Cantor set C defined in Section 3.1 has content zero.
(d) Prove that

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

is integrable, and find the value of the integral.

Solution. (a) Feed the endpoints of the cover into a partition: its subintervals then split into those the cover hides and those on which f is continuous. Let $M > 0$ satisfy $|f| \le M$ on $[a, b]$, let D be the set of discontinuity points, and let $\epsilon > 0$. Choose open intervals $O_1, \dots, O_N$ with $D \subseteq \bigcup O_n$ and $\sum |O_n| < \epsilon/(4M)$, and let

$$P_0 = \{a, b\} \cup \big(\{\text{endpoints of the } O_n\} \cap [a, b]\big),$$

listed as $a = x_0 < x_1 < \cdots < x_m = b$. Since no endpoint of any O_n lies in an open subinterval (x_{k-1}, x_k), each such subinterval is either *bad* — contained in a single O_n — or *good* — disjoint from $\bigcup O_n$, hence disjoint from D.
Bad subintervals have pairwise disjoint interiors, and those sitting inside a fixed O_n have total length at most $|O_n|$, so

$$\sum_{k \text{ bad}} (M_k - m_k)\, \Delta x_k \;\le\; 2M \sum_{n=1}^{N} |O_n| \;<\; \frac{\epsilon}{2}.$$

On a good interval $[x_{k-1}, x_k]$ the restriction of f is bounded and continuous except possibly at the two endpoints, hence integrable by Theorem 7.3.2 together with Theorem 7.4.1 and induction (the remark following Theorem 7.3.2). So Theorem 7.2.8 supplies a partition P_k of $[x_{k-1}, x_k]$ with $U(f, P_k) - L(f, P_k) < \epsilon/(2m)$.
Let P be P_0 together with all the points of these P_k, taking $P_k = \{x_{k-1}, x_k\}$ for bad k. Splitting the sum over bad and good subintervals,

$$U(f, P) - L(f, P) \;<\; \frac{\epsilon}{2} + m \cdot \frac{\epsilon}{2m} \;=\; \epsilon,$$

so f is integrable by Theorem 7.2.8.
(b) For $A = \{a_1, \dots, a_N\}$ take $O_n = \big(a_n - \frac{\epsilon}{4N},\, a_n + \frac{\epsilon}{4N}\big)$. Then $A \subseteq \bigcup O_n$ and $\sum_{n=1}^{N} |O_n| = \epsilon/2 \le \epsilon$.
(c) Recall $C = \bigcap_n C_n$, where C_n is a disjoint union of 2^n closed intervals of length 3^{-n}. Given $\epsilon > 0$, fix n with $(2/3)^n < \epsilon/2$ and set $\delta = \epsilon/2^{n+2}$. Replacing each closed interval $[\alpha, \beta]$ of C_n by the open interval $(\alpha - \delta, \beta + \delta)$ produces 2^n open intervals covering $C_n \supseteq C$ of total length

$$2^n \left(\frac{1}{3^n} + 2\delta\right) = \left(\frac{2}{3}\right)^n + \frac{\epsilon}{2} < \epsilon.$$

(d) $\int_0^1 h = 0$. The set of discontinuities of h is exactly C: if $x \notin C$ then h vanishes on a neighborhood of x (the complement of the closed set C is open), while if $x \in C$ then, because C contains no interval — it lies in C_n, of total length $(2/3)^n \to 0$ — there are points $y \notin C$ arbitrarily close to x, giving $|h(x) - h(y)| = 1$. By (c) this set has content zero, so h is integrable by part (a). Finally, every subinterval of every partition P of $[0, 1]$ contains a point off C, so each $m_k = 0$ and $L(h, P) = 0$; hence

$$\int_0^1 h = L(h) = 0.$$

Problem 7.4.1. Let f be a bounded function on a set A, and set

$$M = \sup\{f(x) : x \in A\}, \qquad m = \inf\{f(x) : x \in A\},$$
$$M' = \sup\{|f(x)| : x \in A\}, \qquad m' = \inf\{|f(x)| : x \in A\}.$$

(a) Show that $M - m \ge M' - m'$.
(b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.
(c) Provide the details for the argument that in this case we have $\left|\int_a^b f\right| \le \int_a^b |f|$.

Solution. (a) Both differences are suprema of difference sets, and $\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|$ does the rest. Precisely, $M - m = \sup f + \sup(-f) = \sup\{f(x) - f(y) : x, y \in A\}$, and likewise

$M' - m' = \sup\{|f(x)| - |f(y)| : x, y \in A\}$. For any $x, y \in A$ the reverse triangle inequality gives

$$|f(x)| - |f(y)| \le |f(x) - f(y)| \le M - m,$$

the last step because both $f(x) - f(y)$ and $f(y) - f(x)$ are at most $M - m$. Taking the supremum over x, y yields $M' - m' \le M - m$.
(b) Let $\epsilon > 0$. Since f is integrable, Theorem 7.2.8 supplies a partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$ with $U(f, P) - L(f, P) < \epsilon$. Write M_k, m_k for the supremum and infimum of f on $[x_{k-1}, x_k]$ and M'_k, m'_k for those of $|f|$. Part (a), applied on each $A = [x_{k-1}, x_k]$, gives $M'_k - m'_k \le M_k - m_k$, so

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{k=1}^{n} (M'_k - m'_k)\,\Delta x_k \\ &\le \sum_{k=1}^{n} (M_k - m_k)\,\Delta x_k = U(f, P) - L(f, P) < \epsilon. \end{aligned}$$

($|f|$ is bounded because f is.) By Theorem 7.2.8, $|f|$ is integrable on $[a, b]$.
(c) By (b) the function $|f|$ is integrable, and $-|f(x)| \le f(x) \le |f(x)|$ for all $x \in [a, b]$. Theorem 7.4.2 (iv) applied to each of these two inequalities, together with Theorem 7.4.2 (ii) with $k = -1$ to evaluate $\int_a^b (-|f|) = -\int_a^b |f|$, gives

$$-\int_a^b |f| \;\le\; \int_a^b f \;\le\; \int_a^b |f|,$$

which is exactly $\left|\int_a^b f\right| \le \int_a^b |f|$.

Problem 7.4.2. (a) Let $g(x) = x^3$, and classify each of the following as positive, negative, or zero.

$$\begin{aligned} &\text{(i)} && \int_0^{-1} g + \int_0^1 g \\ &\text{(ii)} && \int_1^0 g + \int_0^1 g \\ &\text{(iii)} && \int_1^{-2} g + \int_0^1 g. \end{aligned}$$

(b) Show that if $b \le a \le c$ and f is integrable on the interval $[b, c]$, then it is still the case that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Solution. (a) (i) positive, (ii) zero, (iii) positive.
Apply Exercise 7.4.3 (c) to the continuous nonnegative function g on $[0, 1]$ (where $g(1) = 1 > 0$) and to $-g$ on $[-2, 0]$ and on $[-1, 0]$ (where $-g(-1) = 1 > 0$; then $\int(-g) = -\int g$ by Theorem 7.4.2 (ii)):

$$\int_0^1 g > 0, \qquad \int_{-2}^0 g < 0, \qquad \int_{-1}^0 g < 0.$$

(i) $\int_0^{-1} g = -\int_{-1}^0 g > 0$ by Definition 7.4.3, so the sum of two positive numbers is positive.
(ii) $\int_1^0 g = -\int_0^1 g$ by Definition 7.4.3, so the sum is 0.

(iii) By Definition 7.4.3, then Theorem 7.4.1 on $[-2, 1]$ with interior point 0,

$$\begin{aligned}\int_1^{-2} g + \int_0^1 g &= -\int_{-2}^1 g + \int_0^1 g \\ &= -\left(\int_{-2}^0 g + \int_0^1 g\right) + \int_0^1 g \\ &= -\int_{-2}^0 g \;>\; 0.\end{aligned}$$

(b) Assume first $b < a < c$. Theorem 7.4.1, applied to f on $[b, c]$ with the interior point a, gives $\int_b^c f = \int_b^a f + \int_a^c f$. Using Definition 7.4.3 to reverse the orientations,

$$\begin{aligned}\int_a^c f + \int_c^b f &= \int_a^c f - \int_b^c f \\ &= \int_a^c f - \left(\int_b^a f + \int_a^c f\right) \\ &= -\int_b^a f = \int_a^b f.\end{aligned}$$

The degenerate cases are immediate from the convention $\int_c^c f = 0$ of Definition 7.4.3: if $a = b$ both sides equal 0, since $\int_a^c f + \int_c^a f = \int_a^c f - \int_a^c f = 0$; and if $a = c$ the right side is $0 + \int_c^b f = \int_a^b f$.

Problem 7.4.3. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.
(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.
(b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.
(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Solution. (a) False. On $[0, 1]$ take

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then $|f| \equiv 1$ is integrable. But $f = 2g - 1$, where g is Dirichlet's function, so were f integrable then $g = \frac{1}{2}(f+1)$ would be too by Theorem 7.4.2 (i) and (ii) (the constant 1 being integrable), contradicting Example 7.3.3.
(b) False. Take $g = t$, Thomae's function on $[0, 1]$. By Exercise 7.3.2 it is integrable with $\int_0^1 t = 0$, and $t \geq 0$ everywhere while $t(x) > 0$ at every one of the infinitely many rationals in $[0, 1]$.
(c) True. Continuity converts the single positive value into a positive lower bound on a whole subinterval. Put $c = g(y_0) > 0$; g is integrable by Theorem 7.2.9. By continuity at y_0 there is $\delta > 0$ with

$$|g(x) - g(y_0)| < \tfrac{c}{2} \quad \text{for } x \in [a, b],\ |x - y_0| < \delta,$$

so $g(x) > c/2$ for all such x. Since $a < b$, the set $[a, b] \cap (y_0 - \delta, y_0 + \delta)$ contains a closed interval $[\alpha, \beta] \subseteq [a, b]$ with $\alpha < \beta$. Splitting by Theorem 7.4.1 at whichever of α, β is interior to $[a, b]$ (a piece with equal endpoints contributes 0 by Definition 7.4.3) and then bounding each piece by Theorem 7.4.2 (iii), using $g \geq 0$ on the outer two and $g \geq c/2$ on $[\alpha, \beta]$,

$$\begin{aligned}\int_a^b g &= \int_a^\alpha g + \int_\alpha^\beta g + \int_\beta^b g \\ &\geq 0 + \frac{c}{2}(\beta - \alpha) + 0 \;>\; 0.\end{aligned}$$

Problem 7.4.4. Show that if $f(x) > 0$ for all $x \in [a, b]$ and f is integrable, then $\int_a^b f > 0$.

Solution. If the integral were 0, a nested-interval search would produce a point where f vanishes. Theorem 7.4.2 (iii) with $m = 0$ gives $\int_a^b f \geq 0$, so assume for contradiction that $\int_a^b f = 0$.
Construct closed intervals $[a, b] = I_0 \supseteq I_1 \supseteq \cdots$, each of positive length, with

$$\int_{I_n} f = 0 \quad \text{and} \quad \sup_{x \in I_n} f(x) < \tfrac{1}{n} \qquad (n \geq 1);$$

the case $n = 0$ is the contradiction hypothesis. Given $I_{n-1} = [\alpha, \beta]$ with $\alpha < \beta$ and $\int_\alpha^\beta f = 0$, note f is integrable on $[\alpha, \beta]$ by Theorem 7.4.1 and that $\int_\alpha^\beta f$ is the infimum of the upper sums there, so some partition $P = \{\alpha = x_0 < \cdots < x_N = \beta\}$ has $U(f, P) < (\beta - \alpha)/(2n)$. Since $\sum_k \Delta x_k = \beta - \alpha$, the numbers $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ (finite, as f is bounded) have weighted average $U(f, P)/(\beta - \alpha) < 1/(2n)$, so $M_k < 1/(2n)$ for at least one k. Set $I_n = [x_{k-1}, x_k]$: it has positive length, lies in I_{n-1}, and $\sup_{I_n} f = M_k < 1/n$. Splitting at x_{k-1} and x_k (Theorem 7.4.1; a piece with equal endpoints is 0 by Definition 7.4.3),

$$0 = \int_\alpha^\beta f = \int_\alpha^{x_{k-1}} f + \int_{I_n} f + \int_{x_k}^\beta f,$$

whose three terms are all ≥ 0 by Theorem 7.4.2 (iii), forcing $\int_{I_n} f = 0$.
By the Nested Interval Property (Theorem 1.4.1) pick $z \in \bigcap_{n \geq 1} I_n$. Then

$$0 < f(z) < \frac{1}{n} \qquad \text{for every } n \in \mathbb{N},$$

contradicting the Archimedean Property (Theorem 1.4.2 (ii)). Hence $\int_a^b f > 0$.

Problem 7.4.5. Let f and g be integrable functions on $[a, b]$.
(a) Show that if P is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?
(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.

Solution. (a) A supremum of a sum is at most the sum of the suprema. Fix $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and write $M_k(h)$ for the supremum of h on $[x_{k-1}, x_k]$. For every x in that subinterval, $f(x) + g(x) \leq M_k(f) + M_k(g)$, so $M_k(f) + M_k(g)$ is an upper bound for $f + g$ there and hence $M_k(f + g) \leq M_k(f) + M_k(g)$. Multiplying by $\Delta x_k > 0$ and summing,

$$U(f + g, P) = \sum_{k=1}^n M_k(f + g)\, \Delta x_k \leq U(f, P) + U(g, P).$$

For strictness take $[a, b] = [0, 1]$, $f(x) = x$, $g(x) = -x$, and $P = \{0, 1\}$: then $f + g \equiv 0$, so $U(f + g, P) = 0$, while $U(f, P) + U(g, P) = 1 + 0 = 1$.
The same argument with infima (an infimum of a sum is at least the sum of the infima) gives the reversed statement

$$L(f + g, P) \geq L(f, P) + L(g, P).$$

(b) Run the proof of (ii) on a common refinement. The sequential criterion used there (Exercise 7.2.3, the corollary to Theorem 7.2.8) says a bounded h is integrable exactly when some sequence of partitions has $\lim[U(h, P_n) - L(h, P_n)] = 0$, and then $\int_a^b h = \lim U(h, P_n) = \lim L(h, P_n)$.

Take (Q_n) for f and (R_n) for g as supplied by that criterion, and set $P_n = Q_n \cup R_n$. Since P_n refines both, Lemma 7.2.3 gives

$$0 \le U(f, P_n) - L(f, P_n) \le U(f, Q_n) - L(f, Q_n) \longrightarrow 0,$$

and likewise for g; so the single sequence (P_n) computes both integrals. The function $f+g$ is bounded, and by part (a) and its lower-sum analogue,

$$\begin{aligned} L(f, P_n) + L(g, P_n) &\le L(f+g, P_n) \\ &\le U(f+g, P_n) \\ &\le U(f, P_n) + U(g, P_n). \end{aligned}$$

By the Algebraic Limit Theorem (Theorem 2.3.3) the first and last sequences both converge to $\int_a^b f + \int_a^b g$. Hence $U(f+g, P_n) - L(f+g, P_n) \to 0$, so $f+g$ is integrable, and the Squeeze Theorem (Exercise 2.3.3) gives

$$\int_a^b (f+g) = \lim_{n\to\infty} U(f+g, P_n) = \int_a^b f + \int_a^b g.$$

7.4 Exercises 7.4.6–7.5.1

Problem 7.4.6. Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:
(a) If f satisfies $|f(x)| \le M$ on $[a, b]$, show

$$|(f(x))^2 - (f(y))^2| \le 2M|f(x) - f(y)|.$$

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.
(c) Now show that if f and g are integrable, then fg is integrable. (Consider $(f+g)^2$.)

Solution. (a) Factor the difference of squares:

$$\begin{aligned} |(f(x))^2 - (f(y))^2| &= |f(x) - f(y)|\,|f(x) + f(y)| \\ &\le |f(x) - f(y)|\,(|f(x)| + |f(y)|) \\ &\le 2M\,|f(x) - f(y)|. \end{aligned}$$

(b) An integrable function is bounded by definition, so fix $M > 0$ with $|f(x)| \le M$ on $[a, b]$; then f^2 is bounded by M^2. Let $P = \{x_0 < \cdots < x_n\}$ be any partition and write M_k, m_k for the supremum and infimum of f on $[x_{k-1}, x_k]$, and $\widetilde{M}_k$, $\widetilde{m}_k$ for those of f^2. As in Exercise 7.4.1 (a), each of these oscillations is a supremum of differences, so using (a),

$$\begin{aligned} \widetilde{M}_k - \widetilde{m}_k &= \sup\{(f(x))^2 - (f(y))^2\} \\ &\le \sup\{2M|f(x) - f(y)|\} \\ &= 2M\,(M_k - m_k), \end{aligned}$$

the suprema being taken over $x, y \in [x_{k-1}, x_k]$. Multiplying by Δx_k and summing,

$$U(f^2, P) - L(f^2, P) \le 2M\big[U(f, P) - L(f, P)\big].$$

Given $\epsilon > 0$, Theorem 7.2.8 supplies P with $U(f, P) - L(f, P) < \epsilon/(2M)$, and then $U(f^2, P) - L(f^2, P) < \epsilon$. By Theorem 7.2.8 again, f^2 is integrable.
(c) Polarize:

$$fg = \tfrac{1}{2}\Big[(f+g)^2 - f^2 - g^2\Big].$$

By Theorem 7.4.2 (i) the function $f+g$ is integrable, so $(f+g)^2$ is integrable by (b); so are f^2 and g^2. Theorem 7.4.2 (ii) (with $k = -1$) makes $-f^2$ and $-g^2$ integrable, Theorem 7.4.2 (i) makes the sum $(f+g)^2 - f^2 - g^2$ integrable, and Theorem 7.4.2 (ii) (with $k = 1/2$) finishes it. Hence fg is integrable on $[a, b]$.

Problem 7.4.7. Review the discussion immediately preceding Theorem 7.4.4.
(a) Produce an example of a sequence $f_n \to 0$ pointwise on $[0,1]$ where $\lim_{n\to\infty} \int_0^1 f_n$ does not exist.
(b) Produce an example of a sequence g_n with $\int_0^1 g_n \to 0$ but $g_n(x)$ does not converge to zero for any $x \in [0,1]$. To make it more interesting, let's insist that $g_n(x) \geq 0$ for all x and n.

Solution. (a) Alternate the sign of the spike in the text's example. Set

$$f_n(x) = \begin{cases} (-1)^n n & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } x \geq 1/n. \end{cases}$$

This is $(-1)^n$ times the sequence displayed before Theorem 7.4.4, whose integral over $[0,1]$ is 1; by Theorem 7.4.2 (ii),

$$\int_0^1 f_n = (-1)^n,$$

and $((-1)^n)$ diverges, so $\lim_n \int_0^1 f_n$ does not exist. Convergence is still pointwise to 0: $f_n(0) = 0$ for all n, and for fixed $x \in (0,1]$ the Archimedean Property gives N with $1/N < x$, whence $f_n(x) = 0$ for all $n \geq N$.
(b) The typewriter sequence: a bump of shrinking width sweeping repeatedly across $[0,1]$. Each $n \in \mathbb{N}$ has a unique representation $n = 2^k + j$ with $k \geq 0$ an integer and $0 \leq j < 2^k$; define

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \left[\dfrac{j}{2^k}, \dfrac{j+1}{2^k}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Thus $g_1 = 1$ on $[0,1]$; g_2, g_3 are the indicators of $[0,1/2]$ and $[1/2,1]$; $g_4, \dots, g_7$ those of the four quarters; and so on. Each g_n is nonnegative with at most two discontinuities, hence integrable (Theorem 7.3.2 together with Theorem 7.4.1, as in the discussion following Theorem 7.3.2), and

$$\int_0^1 g_n = \frac{1}{2^k}$$

(Check! Bracket each jump by an interval of width δ to get $L(g_n, P) = 2^{-k}$ and $U(g_n, P) \leq 2^{-k} + 2\delta$.) Since $n = 2^k + j < 2^{k+1}$ we get $2^{-k} < 2/n$, so

$$0 \leq \int_0^1 g_n < \frac{2}{n} \longrightarrow 0.$$

Yet $g_n(x) \not\to 0$ for any $x \in [0,1]$: given x and given $k \geq 0$, the dyadic intervals $[j/2^k, (j+1)/2^k]$, $0 \leq j < 2^k$, cover $[0,1]$, so some j has $x \in [j/2^k, (j+1)/2^k]$ and then $g_{2^k+j}(x) = 1$. Hence $g_n(x) = 1$ for infinitely many n, and the sequence $(g_n(x))$ cannot converge to 0.

Problem 7.4.8. For each $n \in \mathbf{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1, \\ 0 & \text{if } 0 \leq x \leq 1/2^n, \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H$.

Solution. $\int_0^1 H = 2/3$.
Each h_n is increasing on $[0,1]$, hence integrable (Exercise 7.2.7), and evaluating on its two pieces gives

$$\int_0^1 h_n = \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) = \frac{1}{2^n} - \frac{1}{4^n}.$$

Since $0 \le h_n(x) \le 1/2^n$ on $[0,1]$ and $\sum 1/2^n$ converges, the Weierstrass M-Test (Corollary 6.4.5) makes the partial sums $H_N = \sum_{n=1}^{N} h_n$ converge to H uniformly on $[0,1]$, and each H_N is integrable by Theorem 7.4.2 (i). So the Integrable Limit Theorem (Theorem 7.4.4) gives that H is integrable with

$$\int_0^1 H = \lim_{N\to\infty} \sum_{n=1}^{N} \int_0^1 h_n = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{4^n} \right)$$
$$= \frac{1/2}{1-1/2} - \frac{1/4}{1-1/4} = 1 - \frac{1}{3} = \frac{2}{3},$$

the first equality again by Theorem 7.4.2 (i).
Method (2): H is the step function equal to 2^{-k} on $(2^{-k-1}, 2^{-k}]$ for each integer $k \ge 0$, with $H(0) = 0$; indeed for such x one has $2^{-n} < x$ exactly when $n \ge k+1$, so $H(x) = \sum_{n \ge k+1} 2^{-n} = 2^{-k}$. Splitting at $2^{-1}, \dots, 2^{-K}$ (Theorem 7.4.1) and discarding the tail via $0 \le \int_0^{2^{-K}} H \le 2^{-K} \to 0$ (Theorem 7.4.2 (iii)),

$$\int_0^1 H = \sum_{k=0}^{\infty} 2^{-k} \left(2^{-k} - 2^{-k-1} \right) = \frac{1}{2} \sum_{k=0}^{\infty} 4^{-k} = \frac{2}{3}.$$

Problem 7.4.9. Let g_n and g be uniformly bounded on $[0,1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \le M$ and $|g_n(x)| \le M$ for all $n \in \mathbf{N}$ and $x \in [0,1]$. Assume $g_n \to g$ pointwise on $[0,1]$ and uniformly on any set of the form $[0,\alpha]$, where $0 < \alpha < 1$.
If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Solution. Given $\epsilon > 0$, fix $\alpha \in (0,1)$ with $2M(1-\alpha) < \epsilon/2$: the crude bound then disposes of $[\alpha, 1]$, while on $[0,\alpha]$ the convergence is uniform.
Each $g_n - g$ is integrable by Theorem 7.4.2 (i), (ii), hence so is $|g_n - g|$ by (v), and all of these are integrable on $[0,\alpha]$ and on $[\alpha,1]$ by Theorem 7.4.1. Those two results give

$$\left| \int_0^1 g_n - \int_0^1 g \right| = \left| \int_0^1 (g_n - g) \right| \le \int_0^\alpha |g_n - g| + \int_\alpha^1 |g_n - g|.$$

On $[\alpha,1]$ we have $|g_n - g| \le |g_n| + |g| \le 2M$, so Theorem 7.4.2 (iii) bounds the second term by $2M(1-\alpha) < \epsilon/2$, for every n. Uniform convergence on $[0,\alpha]$ supplies an N with $|g_n(x) - g(x)| < \epsilon/2$ for all $n \ge N$ and $x \in [0,\alpha]$, and Theorem 7.4.2 (iii) then bounds the first term by $(\epsilon/2)\alpha < \epsilon/2$. Adding,

$$\left| \int_0^1 g_n - \int_0^1 g \right| < \epsilon \qquad (n \ge N),$$

which is exactly $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Problem 7.4.10. Assume g is integrable on $[0,1]$ and continuous at 0. Show

$$\lim_{n\to\infty} \int_0^1 g(x^n)\,dx = g(0).$$

Solution. Apply Exercise 7.4.9 to $g_n(x) = g(x^n)$; its four hypotheses are what needs checking.
Uniform boundedness: g is bounded, being integrable (Definition 7.2.7), say $|g| \le M$ on $[0,1]$, and then $|g_n| \le M$ too.
Pointwise limit: for $x \in [0,1)$ we have $x^n \to 0$ (Example 2.5.3), so $g(x^n) \to g(0)$ by continuity of g at 0, while $g_n(1) = g(1)$ for every n. Thus $g_n \to G$ pointwise, where $G(x) = g(0)$ on $[0,1)$ and $G(1) = g(1)$; note $|G| \le M$. This G is constant on each $[0,\alpha]$ with $\alpha < 1$ and bounded, hence integrable on $[0,1]$ by

Theorem 7.3.2 (the version at the right endpoint), and Theorems 7.4.1 and 7.4.2 (iii), (v) give

$$\int_0^1 G = g(0)\,\alpha + \int_\alpha^1 G, \qquad \left|\int_\alpha^1 G\right| \le M(1-\alpha),$$

so letting $\alpha \to 1^-$ forces $\int_0^1 G = g(0)$.
Uniform convergence on $[0,\alpha]$, $0<\alpha<1$: given $\epsilon>0$, continuity at 0 supplies $\delta>0$ with $|g(t)-g(0)| < \epsilon$ for $0 \le t < \delta$, and $\alpha^n \to 0$ supplies N with $\alpha^N < \delta$; then $x^n \le \alpha^n \le \alpha^N < \delta$ for $n \ge N$ and $x \in [0,\alpha]$, so $|g_n(x) - G(x)| < \epsilon$ throughout $[0,\alpha]$.
Integrability of g_n: let $\epsilon > 0$, pick $\delta \in (0,1)$ with $2M\delta < \epsilon/2$, and set $c = n\delta^{n-1} > 0$. With $\phi(y) = y^n$, the Mean Value Theorem gives $\phi(y_i) - \phi(y_{i-1}) \ge c\,(y_i - y_{i-1})$ whenever $\delta \le y_{i-1} < y_i \le 1$, and ϕ carries $[y_{i-1}, y_i]$ onto $[\phi(y_{i-1}), \phi(y_i)]$, so the oscillation of g_n on the former equals that of g on the latter. Choose a partition Q of $[\delta^n, 1]$ with $U(g,Q) - L(g,Q) < c\,\epsilon/2$ (Theorem 7.2.8, available since g is integrable on $[\delta^n,1]$ by Theorem 7.4.1) and let $P = \{0\} \cup \phi^{-1}(Q)$, a partition of $[0,1]$ whose first subinterval is $[0,\delta]$. Then

$$U(g_n,P) - L(g_n,P) \le 2M\delta + \frac{1}{c}\big[U(g,Q) - L(g,Q)\big] < \epsilon,$$

so g_n is integrable by Theorem 7.2.8.
Exercise 7.4.9 now gives $\lim_{n\to\infty} \int_0^1 g(x^n)\,dx = \int_0^1 G = g(0)$.

Problem 7.4.11. Review the original definition of integrability in Section 7.2, and in particular the definition of the upper integral $U(f)$. One reasonable suggestion might be to bypass the complications introduced in Definition 7.2.7 and simply define the integral to be the value of $U(f)$. Then every bounded function is integrable! Although tempting, proceeding in this way has some significant drawbacks. Show by example that several of the properties in Theorem 7.4.2 no longer hold if we replace our current definition of integrability with the proposal that $\int_a^b f = U(f)$ for every bounded function f.

Solution. Linearity dies: properties (i) and (ii) of Theorem 7.4.2 both fail, and Dirichlet's function does all the work. On $[a,b] = [0,1]$ let

$$d(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

for which $U(d) = 1$ and $L(d) = 0$ (Example 7.3.3).
(i) Additivity fails. Put $f = d$ and $g = 1 - d$, the indicator of the irrationals. Density of the irrationals makes every subinterval of every partition contain a point where $g = 1$, so $U(g,P) = 1$ for every P and hence $U(g) = 1$. But $f+g$ is the constant function 1, whose upper sums all equal 1. Under the proposal,

$$\int_0^1 (f+g) = U(f+g) = 1 \quad \text{while} \quad \int_0^1 f + \int_0^1 g = 1 + 1 = 2.$$

So $\int_0^1 (f+g) \ne \int_0^1 f + \int_0^1 g$; only the inequality $U(f+g) \le U(f) + U(g)$ of Exercise 7.4.5 (a) survives.
(ii) Scalar multiplication fails for every negative k. For $k<0$ one has $U(kf,P) = kL(f,P)$ (proof of Theorem 7.4.2 (ii)), and because $k<0$ an infimum over P on the left is a supremum over P on the right: $U(kf) = kL(f)$. With $k=-1$ and $f=d$,

$$\int_0^1 (-d) = U(-d) = -L(d) = 0 \quad \text{while} \quad (-1)\int_0^1 d = -U(d) = -1.$$

Problem 7.5.1. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Solution. (a) $F(x) = \frac{1}{2}(1 - x^2)$ for $x < 0$ and $F(x) = \frac{1}{2}(1 + x^2)$ for $x \geq 0$; F is continuous and differentiable on all of $\mathbf{R}$, and $F'(x) = f(x)$ everywhere.
For $x < 0$ the integrand is $|t| = -t$ on the interval joining -1 and x, and $-t^2/2$ is an antiderivative there, so Theorem 7.5.1 (i) gives

$$F(x) = \left(-\frac{x^2}{2}\right) - \left(-\frac{1}{2}\right) = \frac{1 - x^2}{2},$$

valid for every $x < 0$ (for $x < -1$ with the orientation convention $\int_{-1}^{x} = -\int_{x}^{-1}$). For $x \geq 0$, Theorem 7.4.1 splits the integral at 0:

$$F(x) = \int_{-1}^{0} (-t)\, dt + \int_{0}^{x} t\, dt = \frac{1}{2} + \frac{x^2}{2}.$$

The two branches agree at 0 (both give 1/2), and each is a polynomial, so F is continuous everywhere. Since f is continuous at every point, Theorem 7.5.1 (ii), applied on an interval $[-R, R]$ with $R > |x|$, makes F differentiable at every x with $F'(x) = |x| = f(x)$.
(b) $F(x) = x + 1$ for $x < 0$ and $F(x) = 2x + 1$ for $x \geq 0$.
Indeed $\int_{-1}^{x} 1\, dt = x + 1$ for $x < 0$, and for $x \geq 0$,

$$F(x) = \int_{-1}^{0} 1\, dt + \int_{0}^{x} 2\, dt = 1 + 2x.$$

Both branches give 1 at $x = 0$, so F is continuous on all of $\mathbf{R}$ (as Theorem 7.5.1 (ii) also guarantees). F is differentiable at every $x \neq 0$, where $F'(x) = f(x)$; at $x = 0$ the left-hand difference quotients tend to 1 and the right-hand ones to 2, so $F'(0)$ does not exist. Thus $F' = f$ exactly on $\mathbf{R} \setminus \{0\}$.

7.5 Exercises 7.5.2–7.5.8

Problem 7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.
(a) If $g = h'$ for some h on $[a, b]$, then g is continuous on $[a, b]$.
(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.
(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

Solution. (a) False. On $[-1, 1]$ set

$$h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

so that h is differentiable everywhere, with $h'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ and

$$h'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0$$

by the squeeze $|x \sin(1/x)| \leq |x|$. Along $x_n = 1/(2\pi n) \to 0$ we get $h'(x_n) \to -1 \neq 0 = h'(0)$, so $g = h'$ is not continuous at 0.
(b) True. g continuous on the compact interval $[a, b]$ is integrable (Theorem 7.2.9), so $h(x) = \int_a^x g$ is defined on $[a, b]$, and Theorem 7.5.1 (ii) applies at every $c \in [a, b]$ (its continuity hypothesis holds at each such c) to give $h'(c) = g(c)$.

(c) False. On $[-1,1]$ with $a=-1$, take

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Every lower sum is 0 (each subinterval contains points where h vanishes) and a partition of mesh δ has $U(h,P) \le 2\delta$, so h is integrable with $H(x) = \int_{-1}^{x} h = 0$ for all x. Then H is differentiable at $c=0$ with $H'(0)=0$, while h is discontinuous at 0.

Problem 7.5.3. The hypothesis in Theorem 7.5.1 (i) that $F'(x) = f(x)$ for all $x \in [a,b]$ is slightly stronger than it needs to be. Carefully read the proof and state exactly what needs to be assumed with regard to the relationship between f and F for the proof to be valid.

Solution. It is enough to assume that F is continuous on $[a,b]$, differentiable on the open interval (a,b), and satisfies $F'(x) = f(x)$ for all $x \in (a,b)$ — no hypothesis on F at the two endpoints beyond continuity.
The proof uses F only through the Mean Value Theorem (Theorem 5.3.2) applied to F on a typical subinterval $[x_{k-1}, x_k]$ of a partition P, and the MVT requires exactly continuity on $[x_{k-1}, x_k]$ and differentiability on (x_{k-1}, x_k). The point t_k it produces lies in the open interval $(x_{k-1}, x_k) \subseteq (a,b)$, so the substitution

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1})$$

only ever invokes the identity $F' = f$ at interior points. The rest of the proof is the estimate $m_k \le f(t_k) \le M_k$, valid since $t_k \in [x_{k-1}, x_k]$, and the telescoping of $\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]$ to $F(b) - F(a)$, which needs only that $F(a)$ and $F(b)$ are the values continuity assigns at the endpoints.

Problem 7.5.4. Show that if $f : [a,b] \to \mathbf{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a,b]$, then $f(x) = 0$ everywhere on $[a,b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Solution. Differentiate the hypothesis. The function $F(x) = \int_a^x f$ is identically zero on $[a,b]$, and f is continuous at every $c \in [a,b]$ (hence integrable, Theorem 7.2.9), so Theorem 7.5.1 (ii) gives

$$f(c) = F'(c) = 0 \qquad \text{for every } c \in [a,b].$$

Continuity is essential. On $[0,1]$ let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Every lower sum is 0 and a partition of mesh δ has $U(f,P) \le 2\delta$, so f is integrable with $\int_0^x f = 0$ for every $x \in [0,1]$, yet $f(1/2) = 1 \ne 0$.

Problem 7.5.5. The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 (the Differentiable Limit Theorem) under the additional assumption that the sequence of derivatives is continuous.
Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a,b]$. Assuming each f_n' is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a,b]$. Show that $g(x) = f'(x)$.

Solution. Pass to the limit in the displayed identity, then differentiate.
First, g is continuous on $[a, b]$: each f_n' is continuous and $f_n' \to g$ uniformly, so the Continuous Limit Theorem (Theorem 6.2.6) applies. In particular g is integrable on $[a, b]$ (Theorem 7.2.9), as is each f_n'.
Fix $x \in [a, b]$. Uniform convergence on $[a, b]$ restricts to uniform convergence on $[a, x]$, so the Integrable Limit Theorem (Theorem 7.4.4) gives $\int_a^x f_n' \to \int_a^x g$. Meanwhile the right side converges by the pointwise hypothesis, so

$$\begin{aligned}\int_a^x g &= \lim_{n\to\infty} \int_a^x f_n' \\ &= \lim_{n\to\infty} \big(f_n(x) - f_n(a)\big) \\ &= f(x) - f(a).\end{aligned}$$

Now g is continuous at every point of $[a, b]$, so Theorem 7.5.1 (ii) says the left-hand side is a differentiable function of x with derivative $g(x)$. The right-hand side differs from f by the constant $f(a)$. Hence f is differentiable on $[a, b]$ and $f'(x) = g(x)$ for all $x \in [a, b]$.

Problem 7.5.6. (Integration-by-parts).
(a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt.$$

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

Solution. (a) Apply Theorem 7.5.1 (i) to $F = hk$. By the product rule (Theorem 5.2.4 (iii)), F is differentiable on $[a, b]$ with

$$F'(t) = h'(t)k(t) + h(t)k'(t) \qquad \text{for all } t \in [a, b],$$

and this is continuous — h and k are continuous because they are differentiable, and h', k' are continuous by hypothesis — hence integrable (Theorem 7.2.9). So F' is an integrable function admitting the antiderivative F on all of $[a, b]$, and Theorem 7.5.1 (i) gives

$$\int_a^b \big(h'(t)k(t) + h(t)k'(t)\big)\,dt = h(b)k(b) - h(a)k(a).$$

Each summand $h'k$ and hk' is itself continuous, hence integrable, so Theorem 7.4.2 (i) splits the left side into $\int_a^b h'k + \int_a^b hk'$. Solving for $\int_a^b hk'$ is the stated formula.
(b) It suffices to assume h and k are differentiable on $[a, b]$ with h' and k' integrable — continuity of the derivatives is not needed.
Continuity of h' and k' was used in (a) only to know that the three functions $h'k$, hk' and their sum are integrable. Under the weaker hypothesis, h and k are still continuous (being differentiable) and therefore integrable by Theorem 7.2.9; Exercise 7.4.6 says the product of two integrable functions is integrable, so $h'k$ and hk' are integrable, and Theorem 7.4.2 (i) makes $(hk)' = h'k + hk'$ integrable with the integral splitting as before. Theorem 7.5.1 (i) then applies verbatim to $F = hk$, whose derivative equals the integrable function $h'k + hk'$ at every point of $[a, b]$, and the same rearrangement yields the formula.

Problem 7.5.7. Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that f is continuous. (To get started, set $G(x) = \int_a^x f$.)

Solution. Set $G(x) = \int_a^x f$: it has the same derivative as the given antiderivative F, so the two differ by a constant and evaluating at a and b yields $\int_a^b f = F(b) - F(a)$. In detail, let $f : [a, b] \to \mathbf{R}$ be continuous and let F satisfy $F'(x) = f(x)$ on $[a, b]$. Then G is defined, since f is continuous on a compact interval and hence integrable (Theorem 7.2.9), and Theorem 7.5.1 (ii) applies at each $c \in [a, b]$ because f is continuous there, giving

$$G'(x) = f(x) = F'(x) \qquad \text{for all } x \in [a, b].$$

Both F and G are differentiable on the interval $[a, b]$ and have the same derivative there, so Corollary 5.3.4 supplies a constant k with

$$G(x) = F(x) + k \qquad \text{for all } x \in [a, b].$$

Evaluating at $x = a$, where $G(a) = \int_a^a f = 0$, identifies $k = -F(a)$. Evaluating at $x = b$ then gives

$$\int_a^b f = G(b) = F(b) - F(a).$$

Problem 7.5.8. (Natural Logarithm and Euler's Constant). Let

$$L(x) = \int_1^x \frac{1}{t}\, dt,$$

where we consider only $x > 0$.
(a) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.
(b) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)
(c) Show $L(x/y) = L(x) - L(y)$.
(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.
(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Solution. (a) $L(1) = 0$, and $L'(x) = 1/x$. Indeed $t \mapsto 1/t$ is continuous, hence integrable, on every closed interval in $(0, \infty)$ (Theorem 7.2.9), so for x in such an interval $[\alpha, \beta]$ we may write $L(x) = G(x) - G(1)$ with $G(x) = \int_\alpha^x dt/t$ (Theorem 7.4.1, valid in any order of the three points by Definition 7.4.3); Theorem 7.5.1 (ii) makes G differentiable at every point of continuity of the integrand, so $L' = G' = 1/x$ there.
(b) Fix $y > 0$ and set $g(x) = L(xy)$ for $x > 0$. By the Chain Rule (Theorem 5.2.5),

$$g'(x) = L'(xy) \cdot y = \frac{1}{xy} \cdot y = \frac{1}{x} = L'(x).$$

By Corollary 5.3.4, $g(x) = L(x) + k$ on the interval $(0, \infty)$. Evaluating at $x = 1$ gives $k = g(1) - L(1) = L(y)$, so

$$L(xy) = L(x) + L(y) \qquad (x, y > 0).$$

(c) Apply (b) to the pair x/y and y:

$$L(x) = L\left(\frac{x}{y} \cdot y\right) = L\left(\frac{x}{y}\right) + L(y),$$

so $L(x/y) = L(x) - L(y)$.
(d) (γ_n) is decreasing and bounded below by 0, so it converges by the Monotone Convergence Theorem (Theorem 2.4.2). Both facts come from the bounds $1/(k+1) \le 1/t \le 1/k$ on $[k, k+1]$, which by Theorem 7.4.2 (iii) (with $m = 1/(k+1)$, $M = 1/k$, and interval length 1) and $L(k+1) - L(k) = \int_k^{k+1} dt/t$ (Theorem 7.4.1) give

$$\frac{1}{k+1} \;\le\; L(k+1) - L(k) \;\le\; \frac{1}{k}.$$

The left inequality yields

$$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \big(L(n+1) - L(n)\big) \le 0,$$

and the right inequality, summed over $k = 1, \ldots, n-1$, yields $L(n) \le \sum_{k=1}^{n-1} 1/k$, whence

$$\gamma_n \ \ge \ \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k} \ = \ \frac{1}{n} \ > \ 0.$$

(e) By (c), $L(2n) - L(n) = L(2)$, so

$$\begin{aligned}\gamma_{2n} - \gamma_n &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} - (L(2n) - L(n)) \\ &= \sum_{k=n+1}^{2n} \frac{1}{k} \ - \ L(2).\end{aligned}$$

The remaining sum is exactly the $2n$-th partial sum s_{2n} of the alternating harmonic series, since subtracting twice the even-indexed terms deletes the first n reciprocals:

$$\begin{aligned}s_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} &= \sum_{k=1}^{2n} \frac{1}{k} - 2\sum_{j=1}^{n} \frac{1}{2j} \\ &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{j=1}^{n} \frac{1}{j} = \sum_{k=n+1}^{2n} \frac{1}{k}.\end{aligned}$$

Hence $s_{2n} = L(2) + (\gamma_{2n} - \gamma_n) \to L(2)$ by (d). The series converges by the Alternating Series Test (Theorem 2.7.7: the terms $1/k$ decrease to 0), so (s_n) has a limit, which its subsequence (s_{2n}) identifies:

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

7.6 Exercises 7.5.9–7.6.4

Problem 7.5.9. Given a function f on $[a, b]$, define the total variation of f to be

$$Vf = \sup \left\{ \sum_{k=1}^{n} \big| f(x_k) - f(x_{k-1}) \big| \right\},$$

where the supremum is taken over all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$.
(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \le \int_a^b |f'|$.
(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

Solution. (a) Telescope the Fundamental Theorem across a partition and pass the absolute value inside. For $P = \{a = x_0 < \cdots < x_n = b\}$, Theorem 7.5.1 (i) applies on each $[x_{k-1}, x_k]$ (f' is continuous there, hence integrable by Theorem 7.2.9, and f is an antiderivative of it), and then Theorem 7.4.2 (v) applies to the integrable function f':

$$\sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \\ \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|,$$

the last equality by repeated use of the additivity of the integral over adjacent intervals (Theorem 7.4.1). The bound is independent of P, so $Vf \leq \int_a^b |f'|$; in particular the supremum is finite.
(b) Fix a partition P as above. Since f is differentiable on $[a, b]$, the Mean Value Theorem (Theorem 5.3.2) supplies $t_k \in (x_{k-1}, x_k)$ with

$$f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}).$$

Writing $m_k = \inf\{|f'(t)| : t \in [x_{k-1}, x_k]\} \leq |f'(t_k)|$, we get

$$\sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right| = \sum_{k=1}^{n} |f'(t_k)| \, (x_k - x_{k-1}) \\ \geq \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = L(|f'|, P).$$

Thus $L(|f'|, P) \leq Vf$ for every partition P, and taking the supremum over P gives

$$\int_a^b |f'| = L(|f'|) = \sup_P L(|f'|, P) \leq Vf,$$

the first equality because $|f'|$ is integrable (Theorem 7.4.2 (v)), so its lower integral is its integral. With (a), $Vf = \int_a^b |f'|$.

Problem 7.5.10. (Change-of-variable Formula). Let $g : [a, b] \to \mathbf{R}$ be differentiable and assume g' is continuous. Let $f : [c, d] \to \mathbf{R}$ be continuous, and assume that the range of g is contained in $[c, d]$ so that the composition $f \circ g$ is properly defined.
(a) Why are we sure f is the derivative of some function? How about $(f \circ g)g'$?
(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(t) \, dt.$$

Solution. (a) Set $F(y) = \int_c^y f$ for $y \in [c, d]$. Since f is continuous on $[c, d]$ it is integrable there (Theorem 7.2.9), and Theorem 7.5.1 (ii) applies at every point of $[c, d]$ (continuity of f is exactly the hypothesis needed), giving $F'(y) = f(y)$ on $[c, d]$. So f is the derivative of F.
For $(f \circ g)g'$, the antiderivative is $F \circ g$. Indeed g is differentiable on $[a, b]$ with range in $[c, d]$, so the Chain Rule (Theorem 5.2.5) gives

$$(F \circ g)'(x) = F'(g(x)) \, g'(x) = f(g(x)) \, g'(x)$$

for all $x \in [a, b]$.
(b) The integrand is continuous on $[a, b]$ – $f \circ g$ by Theorem 4.3.9 (g is differentiable, hence continuous, and f is continuous on a set containing the range of g), g' by hypothesis – hence integrable (Theorem

7.2.9); and by (a), $(F \circ g)' = (f \circ g)g'$ on all of $[a, b]$. Theorem 7.5.1 (i) therefore applies:

$$\begin{aligned}\int_a^b f(g(x))g'(x)\,dx &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_c^{g(b)} f - \int_c^{g(a)} f = \int_{g(a)}^{g(b)} f(t)\,dt,\end{aligned}$$

the last step being the additivity identity $\int_c^{g(a)} f + \int_{g(a)}^{g(b)} f = \int_c^{g(b)} f$, which holds for any three points of $[c, d]$ in any order under the orientation convention of Definition 7.4.3.

Problem 7.5.11. Assume f is integrable on $[a, b]$ and has a "jump discontinuity" at $c \in (a, b)$. This means that both one-sided limits exist as x approaches c from the left and from the right, but that

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x).$$

(This phenomenon is discussed in more detail in Section 4.6.)
(a) Show that, in this case, $F(x) = \int_a^x f$ is not differentiable at $x = c$.
(b) The discussion in Section 5.5 mentions the existence of a continuous monotone function that fails to be differentiable on a dense subset of $\mathbf{R}$. Combine the results of part (a) with Exercise 6.4.10 to show how to construct such a function.

Solution. (a) The two one-sided difference quotients converge to the two one-sided limits of f, which differ. Write $A = \lim_{x \to c^-} f$ and $B = \lim_{x \to c^+} f$, so $A \neq B$; we show

$$\lim_{x \to c^+} \frac{F(x) - F(c)}{x - c} = B, \qquad \lim_{x \to c^-} \frac{F(x) - F(c)}{x - c} = A,$$

which forces the two-sided limit defining $F'(c)$ to fail to exist.
Let $\epsilon > 0$ and pick $\delta > 0$ with $|f(t) - B| < \epsilon$ for all t with $c < t < c + \delta$. Fix $x \in (c, c + \delta)$ and let y satisfy $c < y < x$. On $[y, x] \subset (c, c + \delta)$ we have $B - \epsilon \leq f \leq B + \epsilon$, and f is integrable there (Theorem 7.4.1), so Theorem 7.4.2 (iii) gives

$$(B - \epsilon)(x - y) \;\leq\; \int_y^x f \;\leq\; (B + \epsilon)(x - y).$$

Now $\int_y^x f = F(x) - F(y)$ by Theorem 7.4.1, and F is continuous on $[a, b]$ by Theorem 7.5.1 (ii) (this uses only integrability of f, which is assumed). Letting $y \to c^+$ therefore yields

$$(B - \epsilon)(x - c) \;\leq\; F(x) - F(c) \;\leq\; (B + \epsilon)(x - c),$$

and dividing by $x - c > 0$,

$$\left| \frac{F(x) - F(c)}{x - c} - B \right| \leq \epsilon \qquad \text{for all } x \in (c, c + \delta).$$

The same argument on intervals $[x, y]$ with $x < y < c$ gives the left-hand statement with A in place of B; there $x - c < 0$, so the division reverses the two bounds and again leaves $|(F(x) - F(c))/(x - c) - A| \leq \epsilon$. Since $A \neq B$, the limit $\lim_{x \to c}\bigl(F(x) - F(c)\bigr)/(x - c)$ does not exist, i.e. F is not differentiable at c.
(b) Take $H(x) = \int_0^x h$, where h is the function of Exercise 6.4.10. Explicitly, let $\{r_1, r_2, r_3, \ldots\}$ enumerate $\mathbf{Q}$, put

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n, \\ 0 & \text{for } x \leq r_n, \end{cases} \qquad h(x) = \sum_{n=1}^{\infty} u_n(x),$$

with Definition 7.4.3 giving the meaning of $\int_0^x$ when $x < 0$.

- h is integrable on every $[-M, M]$: by Exercise 6.4.10 it is increasing on $\mathbf{R}$, and increasing functions on a closed interval are integrable (Exercise 7.2.7). It is also bounded, $0 \le h \le \sum 1/2^n = 1$.
- H is continuous on $\mathbf{R}$ by Theorem 7.5.1 (ii), and increasing since $H(y) - H(x) = \int_x^y h \ge 0$ for $x < y$ (Theorem 7.4.1 and Theorem 7.4.2 (iii) with $m = 0$).
- h has a jump discontinuity at each rational r_n, of size exactly $1/2^n$. Split $h = u_n + g_n$ with $g_n = \sum_{m \ne n} u_m$. Each u_m with $m \ne n$ is continuous at r_n (it is locally constant off r_m, and $r_m \ne r_n$), and the partial sums of g_n converge to g_n uniformly by the Weierstrass M-Test (Corollary 6.4.5) with $M_m = 1/2^m$; so g_n is continuous at r_n by the Continuous Limit Theorem (Theorem 6.2.6). Since $\lim_{x \to r_n^-} u_n = 0$ and $\lim_{x \to r_n^+} u_n = 1/2^n$,
$$\lim_{x \to r_n^+} h(x) - \lim_{x \to r_n^-} h(x) = \frac{1}{2^n} \ne 0.$$
- Hence part (a) applies at every r_n: on any $[-M, M]$ containing r_n in its interior, H differs from $x \mapsto \int_{-M}^x h$ by the constant $\int_{-M}^0 h$, so H is not differentiable at any rational.

So H is a continuous, monotone function on $\mathbf{R}$ whose set of nondifferentiability contains the dense set $\mathbf{Q}$.

Problem 7.6.1. Recall from Section 4.1 that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ in lowest terms with } n > 0, \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

is continuous at every irrational and discontinuous at every rational point. Prove that t is integrable on $[0, 1]$ with $\int_0^1 t = 0$. Let $\epsilon > 0$; the strategy, as usual, is to construct a partition P_ϵ of $[0, 1]$ for which $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$.
(a) First, argue that $L(t, P) = 0$ for any partition P of $[0, 1]$.
(b) Consider the set of points $D_{\epsilon/2} = \{x : t(x) \ge \epsilon/2\}$. How big is $D_{\epsilon/2}$?
(c) To complete the argument, explain how to construct a partition P_ϵ of $[0, 1]$ so that $U(t, P_\epsilon) < \epsilon$.

Solution. (a) Every subinterval $[x_{k-1}, x_k]$ of a partition has positive length, hence contains an irrational, where t vanishes; since $t \ge 0$ everywhere, $m_k = \inf\{t(x) : x \in [x_{k-1}, x_k]\} = 0$. Therefore

$$L(t, P) = \sum_{k=1}^n m_k \, \Delta x_k = 0 \quad \text{for every } P,$$

and consequently $L(t) = \sup_P L(t, P) = 0$.
(b) $D_{\epsilon/2}$ is **finite**. Indeed $t(x) \ge \epsilon/2 > 0$ forces x to be rational, and if $x = m/n$ in lowest terms then $t(x) = 1/n \ge \epsilon/2$ means $n \le 2/\epsilon$ (the point $x = 0$, with $t(0) = 1$, is the case $n = 1$). For each such n there are at most $n + 1$ points m/n in $[0, 1]$, so

$$\#D_{\epsilon/2} \;\le\; \sum_{n \le 2/\epsilon} (n + 1) \;<\; \infty.$$

(c) Write $D_{\epsilon/2} = \{d_1, \ldots, d_N\}$ and pick $\delta > 0$ with

$$\delta < \frac{\epsilon}{8N}$$

small enough that the intervals $[d_j - \delta, d_j + \delta]$ are pairwise disjoint. Let P_ϵ consist of 0, 1, and every point $d_j \pm \delta$ lying in $(0, 1)$. The resulting subintervals split into two classes: those of the form $[d_j - \delta, d_j + \delta]$ (truncated at 0 or 1 if necessary), of total length at most $2N\delta < \epsilon/4$, on which we use only the crude bound $M_k \le 1$; and the remaining subintervals, which contain no point of $D_{\epsilon/2}$ and therefore satisfy $M_k \le \epsilon/2$. Hence

$$\begin{aligned} U(t, P_\epsilon) &\le 1 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{2} \sum_{\text{rest}} \Delta x_k \\ &\le \frac{\epsilon}{4} + \frac{\epsilon}{2} \;<\; \epsilon. \end{aligned}$$

With (a), $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$, so t is integrable by the criterion of Theorem 7.2.8, and $\int_0^1 t = L(t) = 0$.

Problem 7.6.2. Let $C \subseteq [0, 1]$ be the Cantor set of Section 3.1, a compact, uncountable set. Define

$$h(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

(a) Show h has discontinuities at each point of C and is continuous at every point of the complement of C. Thus, h is not continuous on an uncountably infinite set.
(b) Now prove that h is integrable on $[0, 1]$.

Solution. Throughout, $C = \bigcap_{n \geq 0} C_n$, where C_n is the union of 2^n pairwise disjoint closed intervals of length 3^{-n}, and consecutive intervals of C_n are separated by open gaps disjoint from $C_n \supseteq C$.
(a) **Points of C.** Let $x \in C$ and $\delta > 0$. Choose n with $3^{-n} < \delta$. Then x lies in one of the closed intervals I of C_n, and $I \subseteq (x - \delta, x + \delta)$ since $|I| = 3^{-n} < \delta$ and $x \in I$. The open middle third of I is deleted at the next stage, so it contains a point $y \notin C$; then

$$|y - x| < \delta \quad \text{and} \quad |h(y) - h(x)| = 1.$$

So no δ works for $\epsilon = 1$: h is discontinuous at x.
Next, **points of C^c.** C is closed, so C^c is open: if $x \notin C$ there is $V_\delta(x) \subseteq C^c$, on which $h \equiv 0$. A locally constant function is continuous at x.
Since C is uncountable (Section 3.1), h is discontinuous on an uncountable set.
(b) $\int_0^1 h = 0$. For the lower sums: C contains no interval (each C_n interval has length $3^{-n} \to 0$), so every subinterval of any partition contains a point off C, giving $m_k = 0$ and $L(h, P) = 0$ for all P.
For the upper sums, fix $\epsilon > 0$ and choose n with $(2/3)^n < \epsilon/2$. Let $I_j = [a_j, b_j]$, $1 \leq j \leq 2^n$, be the intervals of C_n, listed left to right, and pick $\eta > 0$ with

$$2^{n+1}\eta < \frac{\epsilon}{2}$$

small enough that the enlarged intervals $[a_j - \eta,\, b_j + \eta]$ remain pairwise disjoint and inside the gaps. Let P_ϵ be the partition made of 0, 1, and all points $a_j - \eta$, $b_j + \eta$ falling in $(0, 1)$. Its subintervals are of two kinds: the 2^n enlarged intervals, of total length at most $(2/3)^n + 2^{n+1}\eta$, where $M_k \leq 1$; and the intervening intervals $[b_j + \eta,\, a_{j+1} - \eta]$, each contained in an open gap of C_n and hence disjoint from C, where $M_k = 0$. Therefore

$$\begin{aligned} U(h, P_\epsilon) &\leq \left(\tfrac{2}{3}\right)^n + 2^{n+1}\eta \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

so $U(h, P_\epsilon) - L(h, P_\epsilon) < \epsilon$ and h is integrable by Theorem 7.2.8, with $\int_0^1 h = L(h) = 0$.

Problem 7.6.3. Show that any countable set has measure zero.

Solution. Enumerate the set as $A = \{a_1, a_2, a_3, \ldots\}$ and, given $\epsilon > 0$, take

$$O_n = \left(a_n - \frac{\epsilon}{2^{n+2}},\, a_n + \frac{\epsilon}{2^{n+2}}\right), \qquad |O_n| = \frac{\epsilon}{2^{n+1}}.$$

Then $a_n \in O_n$ for every n, so $A \subseteq \bigcup_{n=1}^\infty O_n$, while by the geometric series

$$\begin{aligned} \sum_{n=1}^\infty |O_n| &= \frac{\epsilon}{2} \sum_{n=1}^\infty \frac{1}{2^n} \\ &= \frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

This meets Definition 7.6.1. (A finite set is the case already handled in Example 7.6.2; alternatively, repeat one of its points to produce an infinite enumeration, which changes nothing above.)

Problem 7.6.4. Prove that the Cantor set has measure zero.

Solution. Cover C by the nth stage of its own construction, fattened slightly to make the intervals open. Given $\epsilon > 0$, choose n with $(2/3)^n < \epsilon/2$, and let $I_1, \ldots, I_{2^n}$ be the 2^n closed intervals of length 3^{-n} whose union is C_n. Replace each $I_j = [a_j, b_j]$ by the concentric open interval

$$O_j = \left(a_j - \tfrac{1}{2}3^{-n},\ b_j + \tfrac{1}{2}3^{-n}\right), \qquad |O_j| = 2 \cdot 3^{-n}.$$

Since $C \subseteq C_n = \bigcup_j I_j \subseteq \bigcup_j O_j$ and

$$\sum_{j=1}^{2^n} |O_j| = 2^n \cdot 2 \cdot 3^{-n} = 2\left(\frac{2}{3}\right)^n < \epsilon,$$

Definition 7.6.1 is satisfied with this finite (hence countable) collection. (To have literally infinitely many intervals, append $O_{2^n+k} = (-1, -1 + \epsilon\, 2^{-k-2})$ for $k \geq 1$; these add $\epsilon/4$ to the total, and one starts instead from $(2/3)^n < \epsilon/4$.)

7.7 Exercises 7.6.5–7.6.11

Problem 7.6.5. Show that if two sets A and B each have measure zero, then $A \cup B$ has measure zero as well. In addition, discuss the proof of the stronger statement that the countable union of sets of measure zero also has measure zero. (This second statement is true, but a completely rigorous proof requires a result about double summations discussed in Section 2.8.)

Solution. Halve the budget. Given $\epsilon > 0$, Definition 7.6.1 supplies open intervals $\{O_n\}$ covering A with $\sum_n |O_n| \leq \epsilon/2$ and open intervals $\{P_n\}$ covering B with $\sum_n |P_n| \leq \epsilon/2$. Interleave them into the single countable collection

$$Q_{2n-1} = O_n, \qquad Q_{2n} = P_n.$$

Then $A \cup B \subseteq \bigcup_n Q_n$, and since all terms are nonnegative the partial sums of $\sum |Q_n|$ are bounded by $\sum_n |O_n| + \sum_n |P_n| \leq \epsilon$, so the series converges with

$$\sum_{n=1}^{\infty} |Q_n| \leq \epsilon.$$

For the countable version, let $A = \bigcup_{k=1}^{\infty} A_k$ with each A_k of measure zero, and split ϵ geometrically: choose open intervals $\{O_n^k\}_{n=1}^{\infty}$ with

$$A_k \subseteq \bigcup_{n=1}^{\infty} O_n^k \qquad \text{and} \qquad \sum_{n=1}^{\infty} |O_n^k| \leq \frac{\epsilon}{2^k}.$$

The doubly indexed family $\{O_n^k : k, n \in \mathbf{N}\}$ is a countable union of countable collections, hence countable (Theorem 1.5.8), and it certainly covers A. Fix any enumeration of it as a single sequence $\{Q_m\}$. Every finite partial sum $\sum_{m=1}^{M} |Q_m|$ involves finitely many pairs (k, n), so it is bounded by $\sum_{k=1}^{K} \sum_{n=1}^{\infty} |O_n^k|$ for some K, which is at most $\sum_{k=1}^{\infty} \epsilon\, 2^{-k} = \epsilon$. An increasing bounded sequence of partial sums converges (Monotone Convergence Theorem 2.4.2), so

$$\sum_{m=1}^{\infty} |Q_m| \leq \epsilon.$$

This is exactly the content of Section 2.8: for nonnegative terms the double sum is independent of the order of summation and agrees with either iterated sum, so no rearrangement issue arises.

Problem 7.6.6. Let f be a bounded function on $[a,b]$. Recall (Definition 7.6.3) that f is α-continuous at $x \in [a,b]$ if there exists $\delta > 0$ such that $|f(y) - f(z)| < \alpha$ for all $y, z \in (x-\delta, x+\delta)$, and that

$$D^\alpha = \{x \in [a,b] : f \text{ is not } \alpha\text{-continuous at } x\}.$$

If $\alpha < \alpha'$, show that $D^{\alpha'} \subseteq D^\alpha$.

Solution. α-continuity is the stronger property, so it passes up to α'. If $x \notin D^\alpha$, pick $\delta > 0$ with $|f(y) - f(z)| < \alpha$ for all $y, z \in (x-\delta, x+\delta)$; the same δ gives

$$|f(y) - f(z)| < \alpha < \alpha' \qquad \text{for all } y, z \in (x-\delta, x+\delta),$$

so $x \notin D^{\alpha'}$. Contrapositively, $D^{\alpha'} \subseteq D^\alpha$.

Problem 7.6.7. Let f be a bounded function on $[a,b]$, let D^α be as in Exercise 7.6.6, and let

$$D = \{x \in [a,b] : f \text{ is not continuous at } x\}.$$

(a) Let $\alpha > 0$ be given. Show that if f is continuous at $x \in [a,b]$, then it is α-continuous at x as well. Explain how it follows that $D^\alpha \subseteq D$.
(b) Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now, explain why this guarantees that

$$D = \bigcup_{n=1}^{\infty} D^{\alpha_n} \qquad \text{where } \alpha_n = 1/n.$$

Solution. (a) Run continuity at x with tolerance $\alpha/2$: there is $\delta > 0$ with $|f(y) - f(x)| < \alpha/2$ for all $y \in (x-\delta, x+\delta) \cap [a,b]$. For any two such y, z,

$$|f(y) - f(z)| \le |f(y) - f(x)| + |f(x) - f(z)| < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,$$

which is α-continuity at x (Definition 7.6.3). Thus continuity at x implies $x \notin D^\alpha$; contrapositively $x \in D^\alpha$ forces f to be discontinuous at x, i.e. $D^\alpha \subseteq D$.
(b) Failure of continuity at x means: there is $\alpha_0 > 0$ such that for every $\delta > 0$ some $y \in (x-\delta, x+\delta) \cap [a,b]$ has $|f(y) - f(x)| \ge \alpha_0$. Taking $z = x$ in Definition 7.6.3, no δ witnesses α_0-continuity, so

$$x \in D^{\alpha_0}.$$

Now choose $n \in \mathbf{N}$ with $\alpha_n = 1/n < \alpha_0$ (Archimedean Property, Theorem 1.4.2). By Exercise 7.6.6, $D^{\alpha_0} \subseteq D^{\alpha_n}$, so $x \in D^{\alpha_n}$. Hence $D \subseteq \bigcup_{n=1}^{\infty} D^{\alpha_n}$, and part (a) gives $D^{\alpha_n} \subseteq D$ for each n, so the union is contained in D. The two inclusions give

$$D = \bigcup_{n=1}^{\infty} D^{\alpha_n}.$$

Problem 7.6.8. Prove that for a fixed $\alpha > 0$, the set D^α is closed.
(Context: f is a bounded function on $[a,b]$; by Definition 7.6.3, f is α-continuous at $x \in [a,b]$ if there exists $\delta > 0$ with $|f(y) - f(z)| < \alpha$ for all $y, z \in (x-\delta, x+\delta)$, and D^α is the set of points of $[a,b]$ at which f fails to be α-continuous.)

Solution. D^α contains its limit points, which is exactly what has to be shown (Definition 3.2.7).
Let x be a limit point of D^α and pick $x_n \in D^\alpha$ with $x_n \to x$ (Theorem 3.2.5); since $[a,b]$ is closed, $x \in [a,b]$. Suppose, for contradiction, that f is α-continuous at x, witnessed by some $\delta > 0$. Choose n

with $|x_n - x| < \delta/2$. Then

$$\left(x_n - \tfrac{\delta}{2},\ x_n + \tfrac{\delta}{2}\right) \subseteq (x - \delta,\ x + \delta),$$

so for all $y, z \in (x_n - \delta/2,\ x_n + \delta/2)$ we get $|f(y) - f(z)| < \alpha$. Thus $\delta/2$ witnesses α-continuity of f at x_n, contradicting $x_n \in D^\alpha$.
Hence f fails to be α-continuous at x, i.e. $x \in D^\alpha$.

Problem 7.6.9. Show that there exists a finite collection of disjoint open intervals $\{G_1, G_2, \ldots, G_N\}$ whose union contains D^α and that satisfies

$$\sum_{n=1}^{N} |G_n| < \frac{\epsilon}{4M}.$$

(Context: this is a step in the proof of Lebesgue's Theorem 7.6.5 in the direction ($\Leftarrow$). Here $|f(x)| \leq M$ for all $x \in [a, b]$, the set D of discontinuities of f is assumed to have measure zero, $\epsilon > 0$ is given, and $\alpha = \epsilon/(2(b - a))$.)

Solution. Compactness of D^α turns a countable cover into a finite one, and merging overlaps makes it disjoint.
By Exercise 7.6.7(a), $D^\alpha \subseteq D$, and a subset of a set of measure zero has measure zero (the same cover serves). So by Definition 7.6.1 there are open intervals $O_1, O_2, O_3, \ldots$ with

$$D^\alpha \subseteq \bigcup_{n=1}^{\infty} O_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} |O_n| \leq \frac{\epsilon}{8M}.$$

The set D^α is closed by Exercise 7.6.8 and bounded because $D^\alpha \subseteq [a, b]$, so it is compact; by the open cover characterization in Theorem 7.6.4 (iii) finitely many of them suffice:

$$D^\alpha \subseteq O_{n_1} \cup O_{n_2} \cup \cdots \cup O_{n_m}.$$

Now merge overlaps. Whenever two of these intervals intersect, their union is again a single open interval, of length at most the sum of the two lengths; replacing the pair by that union lowers the count by one, leaves the union of the whole collection unchanged, and does not increase the total length. After at most $m - 1$ such steps the intervals left, call them $G_1, \ldots, G_N$, are pairwise disjoint, still cover D^α, and satisfy

$$\sum_{k=1}^{N} |G_k| \ \leq\ \sum_{i=1}^{m} |O_{n_i}| \ \leq\ \frac{\epsilon}{8M} \ <\ \frac{\epsilon}{4M}.$$

(If $D^\alpha = \emptyset$ the subcover may be empty; take $G_1 = (a - 1,\ a - 1 + \epsilon/(8M))$.)

Problem 7.6.10. Let K be what remains of the interval $[a, b]$ after the open intervals G_n are all removed; that is,

$$K = [a, b] \setminus \bigcup_{n=1}^{N} G_n.$$

Argue that f is uniformly α-continuous on K.
(Recall: f is uniformly α-continuous on a set A if there exists $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \alpha$.)

Solution. K is compact and misses D^α, so this is the α-continuous version of Theorem 4.4.7.
First, K is closed: the union $\bigcup_{n=1}^{N} G_n$ is open, so its complement is closed (Theorem 3.2.13), and K is the intersection of that complement with the closed set $[a, b]$ (Theorem 3.2.14 (ii)). It is bounded, so K is compact by Theorem 7.6.4 (ii). Second, $D^\alpha \subseteq \bigcup_{n=1}^{N} G_n$ by Exercise 7.6.9, so $K \cap D^\alpha = \emptyset$: f is α-continuous at every point of K.

Now imitate Theorem 4.4.7. If f were not uniformly α-continuous on K, then for each $n \in \mathbf{N}$ the choice $\delta = 1/n$ would fail, producing $x_n, y_n \in K$ with

$$|x_n - y_n| < \frac{1}{n} \qquad \text{and} \qquad |f(x_n) - f(y_n)| \geq \alpha.$$

By Theorem 7.6.4 (i) there is a subsequence $x_{n_k} \to x \in K$, and then $y_{n_k} \to x$ as well because $|x_{n_k} - y_{n_k}| \to 0$. Since $x \in K$, f is α-continuous at x: choose $\delta > 0$ with $|f(y) - f(z)| < \alpha$ for all $y, z \in (x - \delta, x + \delta)$. For k large both x_{n_k} and y_{n_k} lie in $(x - \delta, x + \delta)$, whence $|f(x_{n_k}) - f(y_{n_k})| < \alpha$, a contradiction.

Problem 7.6.11. Finish the proof in this direction by explaining how to construct a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$. It will be helpful to break the sum

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{n} (M_k - m_k)\, \Delta x_k$$

into two parts—one over those subintervals that contain points of D^α and the other over subintervals that do not.

Solution. Use the endpoints of the G_n that lie in (a, b), then chop what is left into pieces shorter than the δ of Exercise 7.6.10.
Let $\delta > 0$ be the constant of uniform α-continuity on K from Exercise 7.6.10, and let

$$a = y_0 < y_1 < \cdots < y_p = b$$

list a, b together with every endpoint of every G_n that lies in (a, b). Each open interval (y_{j-1}, y_j) is of exactly one of two types.
(i) (y_{j-1}, y_j) meets some G_n. No endpoint of G_n lies strictly between y_{j-1} and y_j (an endpoint in (a, b) is one of the y_i, and an endpoint outside $[a, b]$ is outside this interval too), so $(y_{j-1}, y_j) \subseteq G_n$. No G_n swallows two of these subintervals, since that would put some y_i with $0 < i < p$ inside G_n, impossible because y_i is an endpoint of some G_m and the G_n are pairwise disjoint open intervals. So the type (i) intervals have total length at most $\sum_n |G_n| < \epsilon/(4M)$ by Exercise 7.6.9.
(ii) (y_{j-1}, y_j) meets no G_n. Then neither endpoint lies in a G_n either, since a G_n containing y_{j-1} or y_j would, being open, also meet the interval; hence $[y_{j-1}, y_j] \subseteq K$.
Let P_ϵ consist of the points y_j together with extra points subdividing each type (ii) interval into pieces of length less than δ; every such piece still lies in K. Now split the sum. On a type (i) subinterval $|f| \leq M$ gives $M_k - m_k \leq 2M$, so

$$\sum_{\text{(i)}} (M_k - m_k)\, \Delta x_k \; < \; 2M \cdot \frac{\epsilon}{4M} \; = \; \frac{\epsilon}{2}.$$

On a type (ii) piece $[x_{k-1}, x_k] \subseteq K$ with $\Delta x_k < \delta$ we get $|f(x) - f(y)| < \alpha$ for all x, y in it, so $M_k - m_k \leq \alpha$ and

$$\begin{aligned} \sum_{\text{(ii)}} (M_k - m_k)\, \Delta x_k \; &\leq \; \alpha \sum_{\text{(ii)}} \Delta x_k \\ &\leq \; \frac{\epsilon}{2(b-a)} \cdot (b - a) \; = \; \frac{\epsilon}{2}. \end{aligned}$$

Adding, $U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$; running the construction with $\epsilon/2$ in place of ϵ makes the difference strictly less than ϵ, so f is integrable by Theorem 7.2.8.

7.8 Exercises 7.6.12–7.6.18

Problem 7.6.12. (a) Prove that D^α has measure zero. Point out that it is possible to choose a cover for D^α that consists of a finite number of open intervals.
(b) Show how this implies that D has measure zero.
(Context: this completes the direction ($\Rightarrow$) of Lebesgue's Theorem 7.6.5. Here f is assumed Riemann-integrable on $[a,b]$, $\epsilon > 0$ is arbitrary, $\alpha > 0$ is fixed, and $P_\epsilon = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition with $U(f,P_\epsilon) - L(f,P_\epsilon) < \alpha\epsilon$.)

Solution. (a) The subintervals of P_ϵ whose interiors meet D^α have total length less than ϵ, because each of them carries an oscillation of at least α.
Let $A = \{k : (x_{k-1}, x_k) \cap D^\alpha \neq \emptyset\}$. Fix $k \in A$ and pick $x \in D^\alpha \cap (x_{k-1}, x_k)$. Since f is not α-continuous at x, every $\delta > 0$ admits $y, z \in (x-\delta, x+\delta)$ with $|f(y) - f(z)| \geq \alpha$; taking δ small enough that $(x-\delta, x+\delta) \subseteq (x_{k-1}, x_k)$ puts such a pair y, z inside the k-th subinterval, so $M_k - m_k \geq \alpha$. Therefore

$$\alpha \sum_{k\in A} \Delta x_k \;\leq\; \sum_{k\in A} (M_k - m_k)\,\Delta x_k \;\leq\; U(f,P_\epsilon) - L(f,P_\epsilon) \;<\; \alpha\epsilon,$$

where the middle inequality holds because every term $(M_k - m_k)\Delta x_k$ is nonnegative. Dividing by α gives $\sum_{k\in A} \Delta x_k < \epsilon$.
Every point of D^α either lies in one of the open subintervals (x_{k-1}, x_k) with $k \in A$ or is one of the $n+1$ partition points, and that finite set is covered by intervals of total length ϵ as in Example 7.6.2. So

$$\begin{aligned} D^\alpha \;\subseteq\; & \bigcup_{k\in A} (x_{k-1}, x_k) \\ & \cup \bigcup_{k=0}^{n} (x_k - \eta,\; x_k + \eta), \qquad \eta = \frac{\epsilon}{2(n+1)}, \end{aligned}$$

a cover by finitely many open intervals of total length less than 2ϵ. As $\epsilon > 0$ was arbitrary (run the argument with $\epsilon/2$), D^α has measure zero in the sense of Definition 7.6.1.
(b) By Exercise 7.6.7(b),

$$D = \bigcup_{n=1}^{\infty} D^{\alpha_n}, \qquad \alpha_n = 1/n,$$

and part (a) applies to each fixed $\alpha_n > 0$. A countable union of sets of measure zero has measure zero (Exercise 7.6.5), so D has measure zero.

Problem 7.6.13. (a) Show that if f and g are integrable on $[a,b]$, then so is the product fg. (This result was requested in Exercise 7.4.6, but notice how much easier the argument is now.)
(b) Show that if g is integrable on $[a,b]$ and f is continuous on the range of g, then the composition $f \circ g$ is integrable on $[a,b]$.

Solution. Both parts are the same two lines: bound the function, then show its discontinuity set sits inside a set of measure zero, and quote Lebesgue's Theorem 7.6.5 in both directions.
(a) Write D_h for the set of discontinuities of a function h on $[a,b]$. Being integrable, f and g are bounded, say $|f| \leq M_1$ and $|g| \leq M_2$, so $|fg| \leq M_1 M_2$ is bounded as well. If f and g are both continuous at x, then so is fg by the Algebraic Continuity Theorem 4.3.4; contrapositively

$$D_{fg} \subseteq D_f \cup D_g.$$

By Theorem 7.6.5 applied to f and to g, the sets D_f and D_g have measure zero, hence so does their union (Exercise 7.6.5), hence so does the subset D_{fg}. Theorem 7.6.5, read in the other direction, makes fg integrable.
(b) Errata: the hypothesis must be that f is continuous on some closed bounded interval $[c,d] \supseteq g([a,b])$, not merely on the range of g; otherwise take $g(x) = x$ on $(0,1]$ with $g(0) = 1$ and $f(t) = 1/t$,

for which $f \circ g$ is unbounded and so not integrable.
Such an interval exists because g is bounded. Since f is continuous on the compact set $[c, d]$, it is bounded there (Extreme Value Theorem 4.4.2), so $f \circ g$ is a bounded function on $[a, b]$. If g is continuous at x, then $f \circ g$ is continuous at x, since f is continuous at the point $g(x) \in [c, d]$ (Theorem 4.3.9); contrapositively

$$D_{f \circ g} \subseteq D_g,$$

which has measure zero by Theorem 7.6.5. A subset of a set of measure zero has measure zero (the same cover works), so $D_{f \circ g}$ has measure zero and $f \circ g$ is integrable by Theorem 7.6.5.

Problem 7.6.14. Let

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(a) Find $g'(0)$.
(b) Use the standard rules of differentiation to compute $g'(x)$ for $x \neq 0$.
(c) Explain why, for every $\delta > 0$, $g'(x)$ attains every value between 1 and -1 as x ranges over the set $(-\delta, \delta)$. Conclude that g' is not continuous at $x = 0$.

Solution. (a) $g'(0) = 0$. The difference quotient is $x \sin(1/x)$ for $x > 0$ and 0 for $x < 0$, so in either case

$$\left| \frac{g(x) - g(0)}{x - 0} \right| \leq |x|,$$

and given $\epsilon > 0$ the choice $\delta = \epsilon$ forces the quotient within ϵ of 0 (Definition 5.2.1).
(b) For $x > 0$, the product rule (Theorem 5.2.4) and the chain rule (Theorem 5.2.5) give

$$\begin{aligned} g'(x) &= 2x \sin(1/x) + x^2 \cos(1/x) \left(-\frac{1}{x^2} \right) \\ &= 2x \sin(1/x) - \cos(1/x), \end{aligned}$$

and $g'(x) = 0$ for $x < 0$, where g is identically zero on a neighborhood.
(c) Fix $\delta > 0$ and choose $n \in \mathbf{N}$ with $n > 1/(2\pi\delta)$, so that both points below lie in $(0, \delta)$. Put

$$u = \frac{1}{(2n+1)\pi} < v = \frac{1}{2n\pi} < \delta.$$

Since $\sin(2n\pi) = \sin((2n+1)\pi) = 0$, the formula in (b) evaluates to

$$g'(v) = -\cos(2n\pi) = -1, \qquad g'(u) = -\cos((2n+1)\pi) = 1.$$

On $[u, v] \subseteq (0, \delta)$ the function g' is continuous, being built from x, $\sin(1/x)$ and $\cos(1/x)$ by the algebraic and compositional rules (Theorems 4.3.4 and 4.3.9) away from 0. So the Intermediate Value Theorem 4.5.1 hands us every value between -1 and 1 as x ranges over $[u, v]$, hence over $(-\delta, \delta)$.
In particular, taking $\epsilon_0 = 1/2$: for every $\delta > 0$ there is a point $u \in (-\delta, \delta)$ with

$$|g'(u) - g'(0)| = |1 - 0| = 1 \geq \epsilon_0,$$

so g' fails the ϵ-δ definition of continuity (Definition 4.3.1) at $x = 0$.

Problem 7.6.15. (Context.) Let $C = \bigcap_{n=0}^{\infty} C_n$ be the Cantor set of Section 3.1, and let

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Define $f_0(x) = 0$ on $C_0 = [0,1]$. Given f_{n-1}, define f_n by leaving it equal to f_{n-1} off the 2^{n-1} open intervals removed from C_{n-1} to form C_n, and, on each such removed interval (a,b), placing translated copies of g oscillating toward the two endpoints and splicing them together in the middle so that f_n is differentiable and

$$|f_n(x)| \leq (x-a)^2 \quad \text{and} \quad |f_n(x)| \leq (b-x)^2$$

there. (Thus $f_n = 0$ on C_n, and f_n is differentiable on $[0,1]$ with f_n' discontinuous exactly at the endpoints of the intervals making up C_n.) This yields a sequence $f_0, f_1, f_2, \ldots$ defined on $[0,1]$.
(a) If $c \in C$, what is $\lim_{n\to\infty} f_n(c)$?
(b) Why does $\lim_{n\to\infty} f_n(x)$ exist for $x \notin C$?
Now, set

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Solution. (a) $\lim_{n\to\infty} f_n(c) = 0$, because the limit is of the constant sequence 0: if $c \in C$ then $c \in C_n$ for every n, and each f_n vanishes identically on C_n.
(b) The sequence $(f_n(x))$ is eventually constant. Since $x \notin C = \bigcap_n C_n$, there is a first N with $x \notin C_N$, so x lies in one of the 2^{N-1} open intervals I removed from C_{N-1} at stage N. For each $n > N$ the function f_n differs from f_{n-1} only on the open intervals removed from C_{n-1}, and those lie inside $C_{n-1} \subseteq C_N$, which is disjoint from I. Hence

$$f_n(x) = f_N(x) \qquad \text{for all } n \geq N,$$

and the limit exists and equals $f_N(x)$. The same N serves every point of I, so in fact $f = f_N$ on all of I.

Problem 7.6.16. (With $f(x) = \lim_{n\to\infty} f_n(x)$ the function built in Exercise 7.6.15 from the Cantor set C and the function $g(x) = x^2 \sin(1/x)$ for $x > 0$, $g(x) = 0$ for $x \leq 0$.)
(a) Explain why $f'(x)$ exists for all $x \notin C$.
(b) If $c \in C$, argue that $|f(x)| \leq (x-c)^2$ for all $x \in [0,1]$. Show how this implies $f'(c) = 0$.
(c) Give a careful argument for why $f'(x)$ fails to be continuous on C. Remember that C contains many points besides the endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$.

Solution. (a) Because f coincides with a differentiable function on a neighborhood of x. If $x \notin C$, then by Exercise 7.6.15(b) x lies in some open interval I removed at stage N, and $f = f_N$ on all of I. Differentiability is a local property, so $f'(x) = f_N'(x)$, which exists since each f_N was constructed to be differentiable on $[0,1]$.
(b) If $x \in C$ then $f(x) = 0$ and the bound is trivial. If $x \notin C$, let (a,b) be the open interval removed at stage N that contains x; the construction gives

$$|f(x)| \leq (x-a)^2 \quad \text{and} \quad |f(x)| \leq (b-x)^2.$$

Since $(a,b) \cap C = \emptyset$ and $c \in C$, either $c \leq a$ or $c \geq b$:
- (i) $c \leq a$: then $|x-c| = x-c \geq x-a > 0$, so $(x-c)^2 \geq (x-a)^2 \geq |f(x)|$.
- (ii) $c \geq b$: then $|x-c| = c-x \geq b-x > 0$, so $(x-c)^2 \geq (b-x)^2 \geq |f(x)|$.

In either case $|f(x)| \leq (x-c)^2$. Now $f(c) = 0$, so for $x \neq c$,

$$\left|\frac{f(x)-f(c)}{x-c}\right| = \frac{|f(x)|}{|x-c|} \leq |x-c|,$$

and the squeeze gives $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$.
(c) Every neighborhood of every $c \in C$ contains points where f' equals 1, while $f'(c) = 0$ by (b). Fix $c \in C$ and $\delta > 0$, and choose n with $3^{-n} < \delta$. The set C_n is a union of 2^n closed intervals of length 3^{-n}, and $c \in C \subseteq C_n$ lies in one of them, say J; since $|J| < \delta$ and $c \in J$, we have $J \subseteq (c-\delta, c+\delta)$.

The middle third $I = (a, b)$ of J is one of the intervals removed at stage $n + 1$, so $I \subseteq (c - \delta, c + \delta)$, and by Exercise 7.6.15(b) $f = f_{n+1}$ on I. Just to the right of the left endpoint a the construction puts an untranslated copy of g:

$$f(a + t) = g(t) \qquad \text{for all sufficiently small } t > 0.$$

Hence $f'(a+t) = g'(t) = 2t \sin(1/t) - \cos(1/t)$ for such t. Taking $t_k = 1/((2k+1)\pi)$ with k large gives $t_k \to 0$ and

$$f'(a + t_k) = 2t_k \sin\big((2k + 1)\pi\big) - \cos\big((2k + 1)\pi\big) = 1.$$

Taking $\delta = 1/k$ and such a point x_k gives $x_k \to c$ with $f'(x_k) = 1 \not\to 0 = f'(c)$, so f' is discontinuous at c by the sequential criterion (Theorem 4.3.2). The only property of c used was $c \in C_n$ for every n, so this covers all of C, not just the endpoints of the removed intervals.

Problem 7.6.17. (With f the differentiable function of Exercises 7.6.15 and 7.6.16, whose derivative is discontinuous at every point of the Cantor set C.) Why is $f'(x)$ Riemann-integrable on $[0, 1]$?

Solution. Because f' is bounded and its set of discontinuities is contained in C, a set of measure zero; Lebesgue's Theorem 7.6.5 then gives integrability.
Boundedness: $f' = 0$ on C by Exercise 7.6.16(b), and on a removed interval (a, b) we have $f(a+t) = g(t)$ and $f(b - t) = g(t)$ for small $t > 0$, where

$$|g'(t)| = |2t \sin(1/t) - \cos(1/t)| \leq 2t + 1 \leq 3.$$

The splice left unspecified in the text may be taken constant: join the two copies at a common s with $g'(s) = 0$ (such s occur arbitrarily near 0) and set $f = g(s)$ in between. This is differentiable, keeps $|f| \leq s^2 \leq \min\{(x - a)^2, (b - x)^2\}$ there, and contributes $f' = 0$. Hence $|f'| \leq 3$ on $[0, 1]$.
Discontinuity set: if $x \notin C$ then $f = f_N$ on a whole open interval about x (Exercise 7.6.15(b)), and on that interval f' is $g'(t)$ with $t > 0$ near the endpoints and 0 in the middle, hence continuous at x. So the discontinuity set of f' is contained in C, which has measure zero by Exercise 7.6.4; a subset of a measure-zero set is covered by the same intervals, so it too has measure zero. Theorem 7.6.5 applies.

Problem 7.6.18. The reason the Cantor set has measure zero is that, at each stage, 2^{n-1} open intervals of length $1/3^n$ are removed from C_{n-1}. The resulting sum

$$\sum_{n=1}^{\infty} 2^{n-1} \left(\frac{1}{3^n}\right)$$

converges to one, which means that the approximating sets $C_1, C_2, C_3, \ldots$ have total lengths tending to zero. Instead of removing open intervals of length $1/3^n$ at each stage, let us see what happens when we remove intervals of length $1/3^{n+1}$.
Show that, under these circumstances, the sum of the lengths of the intervals making up each C_n no longer tends to zero as $n \to \infty$. What is this limit?

Solution. The limit is 2/3.

At stage n we delete 2^{n-1} open intervals of length $1/3^{n+1}$, so the length of C_n is

$$\begin{aligned}|C_n| &= 1 - \sum_{k=1}^{n} 2^{k-1}\left(\frac{1}{3^{k+1}}\right)\\ &= 1 - \frac{1}{9}\sum_{k=1}^{n}\left(\frac{2}{3}\right)^{k-1}\\ &= 1 - \frac{1}{9}\cdot\frac{1-(2/3)^n}{1-2/3}\\ &= \frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^n.\end{aligned}$$

(There is always room: the 2^{n-1} intervals of C_{n-1} are congruent, of length $2^{-(n-1)}|C_{n-1}| \geq 2^{-(n-1)}\cdot\frac{2}{3} = 4(3/2)^n\cdot 3^{-(n+1)} > 3^{-(n+1)}$.)
Since $(2/3)^n \to 0$,

$$\lim_{n\to\infty}|C_n| = \frac{2}{3} \neq 0.$$

7.9 Exercises 7.6.19–7.6.19

Problem 7.6.19. As a final gesture, provide the example advertised in Exercise 7.6.13 of an integrable function f and a continuous function g where the composition $f\circ g$ is properly defined but not integrable. Exercise 4.3.12 may be useful.
(Exercise 7.6.13(b) showed that if g is integrable on $[a,b]$ and f is continuous on the range of g, then $f\circ g$ is integrable; the point here is that reversing the hypotheses destroys the conclusion. Exercise 4.3.12 states that for a nonempty closed set $F\subseteq\mathbf{R}$ the function $x\mapsto\inf\{|x-a| : a\in F\}$ is continuous on all of $\mathbf{R}$ and is nonzero off F.)

Solution. Take $D = \bigcap_{n=0}^{\infty} C_n$ the Cantor-type set of Exercise 7.6.18, and set

$$f(y) = \begin{cases}1 & \text{if } y = 0,\\ 0 & \text{if } y \neq 0,\end{cases} \qquad g(x) = \inf\{|x-a| : a\in D\},$$

both on $[0,1]$. Then $f\circ g$ is the characteristic function of D, which is not integrable.
Integrability of f: it is bounded, and its only discontinuity is $y=0$; a one-point set has measure zero (Example 7.6.2), so Theorem 7.6.5 applies.
Continuity of g, and the composition: D is nonempty and closed (an intersection of finite unions of closed intervals), so Exercise 4.3.12 gives continuity of g on $\mathbf{R}$, together with $g(x)\neq 0$ for $x\notin D$; and $g(x)=0$ for $x\in D$. Since $0\in D\subseteq[0,1]$, every $x\in[0,1]$ has $0\leq g(x)\leq |x-0|\leq 1$, so $g([0,1])\subseteq[0,1]$ and $f\circ g$ is properly defined there. Hence

$$(f\circ g)(x) = \begin{cases}1 & \text{if } x\in D,\\ 0 & \text{if } x\notin D.\end{cases}$$

The discontinuity set of $f\circ g$ is exactly D. Indeed, if $x\notin D$, then $[0,1]\setminus D$ is open and $f\circ g$ vanishes on a neighborhood of x, so $f\circ g$ is continuous at x. If $x\in D$, given $\epsilon>0$ choose n with $2^{-n}<\epsilon$; the component J of C_n containing x has length $|C_n|/2^n\leq 2^{-n}<\epsilon$, so $J\subseteq(x-\epsilon,x+\epsilon)$, and any y in the open middle removed from J at stage $n+1$ has $y\notin D$ and $|(f\circ g)(y)-(f\circ g)(x)| = 1$. So $f\circ g$ is discontinuous at x.
Finally, D does not have measure zero. Suppose $\{O_k\}_{k=1}^{\infty}$ is a countable cover of D by open intervals with $\sum_k |O_k| < 1/2$. The set D is closed and bounded, hence compact (Theorem 3.3.4), so finitely many of the O_k cover D; let U be their union, an open set with total length less than $1/2$. Because $\bigcap_n C_n = D\subseteq U$, the relatively open sets U and $[0,1]\setminus C_1\subseteq[0,1]\setminus C_2\subseteq\cdots$ cover the compact set

$[0,1]$, so a finite subcollection does, and as the C_n are nested this gives

$$\begin{aligned} [0,1] &\subseteq U \cup \big([0,1] \setminus C_n\big), \\ \text{i.e.} \quad C_n &\subseteq U, \end{aligned}$$

for some n. But C_n is a finite union of disjoint closed intervals of total length $|C_n| \geq 2/3$ (Exercise 7.6.18), total length is monotone on finite unions of intervals, so $C_n \subseteq U$ forces $2/3 \leq |C_n| < 1/2$ – a contradiction. Hence no such cover exists.
Thus $f \circ g$ is bounded and its set of discontinuities does not have measure zero, so Theorem 7.6.5 denies it Riemann-integrability on $[0,1]$.

8 Additional Topics

Contents

8.1 Exercises 8.1.1–8.1.7

Problem 8.1.1. (Context: the forward direction of the proof of Theorem 8.1.2, the Limit Criterion for Riemann Integrability. We assume f is integrable and bounded by $M > 0$ on $[a, b]$, we have fixed a partition P_ϵ with $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon/3$ whose number of subintervals is n, we have set $\delta = \epsilon/9nM$, and $(P, \{c_k\})$ is an arbitrary δ-fine tagged partition with $P' = P \cup P_\epsilon$.)
(a) Explain why both the Riemann sum $R(f, P)$ and $\int_a^b f$ fall between $L(f, P)$ and $U(f, P)$.
(b) Explain why $U(f, P') - L(f, P') < \epsilon/3$.

Solution. (a) Both are trapped between the same two sums. Each tag lies in its subinterval, so $m_k \leq f(c_k) \leq M_k$; multiplying by $x_k - x_{k-1} > 0$ and summing over k gives

$$L(f, P) \leq R(f, P) \leq U(f, P).$$

For the integral, Definition 7.2.7 makes $\int_a^b f$ the common value of $L(f) = \sup_Q L(f, Q)$ and $U(f) = \inf_Q U(f, Q)$, and P is one competing partition in each:

$$L(f, P) \leq L(f) = \int_a^b f = U(f) \leq U(f, P).$$

(b) $P' = P \cup P_\epsilon \supseteq P_\epsilon$, so Lemma 7.2.3 (refining raises lower sums and lowers upper sums) gives

$$U(f, P') - L(f, P') \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon/3,$$

the last step being the defining property of P_ϵ.

Problem 8.1.2. (Continuing the proof of Theorem 8.1.2, with P a δ-fine partition of $[a, b]$ and $P' = P \cup P_\epsilon$.) Explain why $U(f, P) - U(f, P') \geq 0$.

Solution. Because $P \subseteq P \cup P_\epsilon = P'$, so P' refines P and Lemma 7.2.3 gives $U(f, P) \geq U(f, P')$ at once. The mechanism is worth recording, since the next exercise counts terms: inserting one point z into $[x_{k-1}, x_k]$ replaces $M_k(x_k - x_{k-1})$ by

$$M_k'(z - x_{k-1}) + M_k''(x_k - z) \leq M_k(x_k - x_{k-1}),$$

since M_k' and M_k'' are suprema of f over subsets of $[x_{k-1}, x_k]$ and so are $\leq M_k$. Passing from P to P' is a finite succession of such insertions.

Problem 8.1.3. (Continuing the proof of Theorem 8.1.2: $|f| \leq M$, the partition P_ϵ has n subintervals, $\delta = \epsilon/9nM$, P is δ-fine, and $P' = P \cup P_\epsilon$.)
(a) In terms of n, what is the largest number of terms of the form $M_k(x_k - x_{k-1})$ that could appear in one of $U(f, P)$ or $U(f, P')$ but not the other?
(b) Finish the proof in this direction by arguing that

$$U(f, P) - U(f, P') < \epsilon/3.$$

Solution. (a) At most $3(n-1)$, namely at most $n-1$ from $U(f, P)$ and at most $2(n-1)$ from $U(f, P')$. Indeed P_ϵ has n subintervals, hence $n+1$ points, of which a and b already lie in P; so P' is P with at most $n-1$ points inserted. By Exercise 8.1.2 each insertion kills one term of $U(f, P)$ and creates two of $U(f, P')$, while every untouched subinterval contributes the same term to both sums and cancels. (Two insertions into one subinterval of P only lower the count.)
(b) Each surviving term is $M_j\,\Delta_j$ with $\Delta_j < \delta$ (every subinterval of P or of the refinement P' sits inside a subinterval of the δ-fine P) and $|M_j| \leq M$ (as $|f| \leq M$). With $U(f, P) - U(f, P') \geq 0$ from

Exercise 8.1.2 and the triangle inequality,

$$\begin{aligned}0 \le U(f,P) - U(f,P') &\le 3(n-1)\,M\,\delta \\ &< 3n\,M \cdot \frac{\epsilon}{9nM} = \frac{\epsilon}{3}.\end{aligned}$$

The same argument on lower sums gives $L(f,P') - L(f,P) < \epsilon/3$, so the promised string

$$L(f,P') - \frac{\epsilon}{3} < L(f,P) \le U(f,P) < U(f,P') + \frac{\epsilon}{3}$$

holds, and with $U(f,P') - L(f,P') < \epsilon/3$ from Exercise 8.1.1(b),

$$U(f,P) - L(f,P) < \big(U(f,P') - L(f,P')\big) + \tfrac{2\epsilon}{3} < \epsilon.$$

By Exercise 8.1.1(a) both $R(f,P)$ and $\int_a^b f$ lie in $[L(f,P), U(f,P)]$, so

$$\left| R(f,P) - \int_a^b f \right| < \epsilon$$

for every δ-fine tagged partition $(P, \{c_k\})$.

Problem 8.1.4. (Setting up the converse direction of Theorem 8.1.2. Here $f : [a,b] \to \mathbf{R}$ is bounded and $P = \{x_0, x_1, \ldots, x_n\}$ is a fixed partition of $[a,b]$.)
(a) Show that if f is continuous, then it is possible to pick tags $\{c_k\}_{k=1}^n$ so that

$$R(f,P) = U(f,P).$$

Similarly, there are tags for which $R(f,P) = L(f,P)$ as well.
(b) If f is not continuous, it may not be possible to find tags for which $R(f,P) = U(f,P)$. Show, however, that given an arbitrary $\epsilon > 0$, it is possible to pick tags for P so that

$$U(f,P) - R(f,P) < \epsilon.$$

The analogous statement holds for lower sums.

Solution. (a) Tag each subinterval at a maximum point of f there. Such a c_k exists because f is continuous on the compact set $[x_{k-1}, x_k]$, so the Extreme Value Theorem (Theorem 4.4.2) gives c_k with $f(c_k) = M_k$, and

$$R(f,P) = \sum_{k=1}^n f(c_k)(x_k - x_{k-1}) = \sum_{k=1}^n M_k(x_k - x_{k-1}) = U(f,P).$$

Tagging at minimum points instead gives $f(c_k) = m_k$ and $R(f,P) = L(f,P)$.
(b) Tag with $c_k \in [x_{k-1}, x_k]$ satisfying $f(c_k) > M_k - \dfrac{\epsilon}{b-a}$, which Lemma 1.3.8 supplies: M_k is finite (f is bounded) and $M_k - \epsilon/(b-a) < M_k$ is therefore not an upper bound for the values of f on $[x_{k-1}, x_k]$. Since $\sum_{k=1}^n (x_k - x_{k-1}) = b - a$,

$$\begin{aligned}U(f,P) - R(f,P) &= \sum_{k=1}^n \big(M_k - f(c_k)\big)(x_k - x_{k-1}) \\ &< \frac{\epsilon}{b-a} \sum_{k=1}^n (x_k - x_{k-1}) = \epsilon.\end{aligned}$$

For lower sums, pick $d_k \in [x_{k-1}, x_k]$ with $f(d_k) < m_k + \epsilon/(b-a)$ by the infimum form of Lemma 1.3.8 (Exercise 1.3.1(b)); writing $R'(f,P)$ for the Riemann sum with tags $\{d_k\}$, the same computation gives $R'(f,P) - L(f,P) < \epsilon$.

Problem 8.1.5. Use the results of the previous exercise to finish the proof of Theorem 8.1.2. That is, assuming $f : [a, b] \to \mathbf{R}$ is bounded and that for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-fine tagged partition $(P, \{c_k\})$ satisfies $|R(f, P) - A| < \epsilon$, prove that f is Riemann-integrable with $\int_a^b f = A$.

Solution. Squeeze $U(f, P)$ and $L(f, P)$ against A using the two sets of tags from Exercise 8.1.4(b). Given $\epsilon > 0$, take the $\delta > 0$ the hypothesis provides for $\epsilon/4$, and let P be the uniform partition with $N > (b-a)/\delta$ subintervals (N exists by the Archimedean Property, Theorem 1.4.2), so P is δ-fine. By Exercise 8.1.4(b) pick tags $\{c_k\}$ with $U(f, P) - R(f, P) < \epsilon/4$ and tags $\{d_k\}$ with $R'(f, P) - L(f, P) < \epsilon/4$, where $R'(f, P)$ is the Riemann sum over P tagged by $\{d_k\}$. Both tagged partitions are δ-fine, so both sums lie within $\epsilon/4$ of A, and

$$\begin{aligned} U(f, P) - L(f, P) &= \big(U(f, P) - R(f, P)\big) + \big(R(f, P) - A\big) \\ &\quad + \big(A - R'(f, P)\big) + \big(R'(f, P) - L(f, P)\big) \\ &< 4 \cdot \tfrac{\epsilon}{4} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, f is integrable by the Integrability Criterion (Theorem 7.2.8).
For the value: Exercise 8.1.1(a) places both $\int_a^b f$ and $R(f, P)$ in $[L(f, P), U(f, P)]$, an interval of length less than ϵ, so

$$\begin{aligned} \left| A - \int_a^b f \right| &\le |A - R(f, P)| + \left| R(f, P) - \int_a^b f \right| \\ &< \tfrac{\epsilon}{4} + \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, $\int_a^b f = A$.

Problem 8.1.6. Consider the interval $[0, 1]$.
(a) If $\delta(x) = 1/9$, find a $\delta(x)$-fine tagged partition of $[0, 1]$. Does the choice of tags matter in this case?
(b) Let

$$\delta(x) = \begin{cases} 1/4 & \text{if } x = 0, \\ x/3 & \text{if } 0 < x \le 1. \end{cases}$$

Construct a $\delta(x)$-fine tagged partition of $[0, 1]$.

Solution. (a) Take $P = \{0, \frac{1}{10}, \frac{2}{10}, \ldots, 1\}$ with any tags at all, say $c_k = x_k$: every width is $1/10 < 1/9 = \delta(c_k)$. No, the tags do not matter, because for a constant gauge the condition $x_k - x_{k-1} < \delta(c_k)$ reads $x_k - x_{k-1} < 1/9$ and never mentions c_k; Definition 8.1.4 collapses to Definition 8.1.1.
(b) Take

$$P = \{0\} \cup \big\{ \tfrac{k}{20} : k = 4, 5, \ldots, 20 \big\},$$

tagged by $c_1 = 0$ and $c_k = x_k$ for $k \ge 2$. The first subinterval has width $\frac{1}{5} < \frac{1}{4} = \delta(0)$; every later one has width $\frac{1}{20}$ and tag $c_k \ge \frac{1}{5}$, so

$$\tfrac{1}{20} < \tfrac{1}{15} \le \tfrac{c_k}{3} = \delta(c_k).$$

The tag $c_1 = 0$ is forced, and this is the tinkering the section alludes to: for $c_1 \in (0, x_1]$ we would have $\delta(c_1) = c_1/3 \le x_1/3$, and fineness would demand $x_1 < x_1/3$, impossible.

Problem 8.1.7. Finish the proof of Theorem 8.1.5: given a gauge $\delta(x)$ on an interval $[a, b]$, there exists a tagged partition $(P, \{c_k\}_{k=1}^n)$ that is $\delta(x)$-fine. (The proof begins by setting $I_0 = [a, b]$; if $b - a < \delta(x)$ for some $x \in [a, b]$ then the trivial partition $P = \{a, b\}$ with $c_1 = x$ works, and if no such x exists then $[a, b]$ is bisected into two equal halves.)

Solution. Bisect and apply the Nested Interval Property; the offending point in the intersection supplies its own tag.

Call a closed interval $I \subseteq [a, b]$ *good* if I admits a $\delta(x)$-fine tagged partition, and assume for contradiction that $I_0 = [a, b]$ is not good.

Goodness glues: if $[u, c]$ and $[c, w]$ are both good, so is $[u, w]$, because concatenating the two tagged partitions gives a tagged partition of $[u, w]$ and fineness, $x_k - x_{k-1} < \delta(c_k)$, is a condition on each subinterval separately.

So at least one half of I_0 is not good; call it I_1, and repeat, obtaining nested intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots, \qquad |I_m| = \frac{b-a}{2^m},$$

with no I_m good. By the Nested Interval Property (Theorem 1.4.1) there exists

$$x_0 \in \bigcap_{m=0}^{\infty} I_m.$$

Since δ is a gauge, $\delta(x_0) > 0$, so by the Archimedean Property (Theorem 1.4.2) there is an $m \in \mathbf{N}$ with

$$|I_m| = \frac{b-a}{2^m} < \delta(x_0)$$

(take $m > (b-a)/\delta(x_0)$ and use $2^m \geq m$). But $x_0 \in I_m$, so the trivial partition of $I_m = [u, w]$ by its endpoints, tagged with $c_1 = x_0$, has $w - u = |I_m| < \delta(x_0) = \delta(c_1)$ and is $\delta(x)$-fine. Thus I_m is good, a contradiction, and $[a, b]$ is good.

8.2 Exercises 8.1.8–8.1.14

Problem 8.1.8. This exercise completes the proof of Theorem 8.1.7: *if a function has a generalized Riemann integral, then the value of the integral is unique.* Abbott's proof opens as follows. Assume that a function f has generalized Riemann integral A_1 and that it also has generalized Riemann integral A_2. We must prove $A_1 = A_2$. Finish the argument.

Solution. Take the pointwise minimum of the two gauges and feed it to Theorem 8.1.5.

Let $\epsilon > 0$. By Definition 8.1.6 there are gauges $\delta_1(x)$ and $\delta_2(x)$ on $[a, b]$ such that

$$\left|R(f, P) - A_i\right| < \frac{\epsilon}{2}$$

whenever $(P, \{c_k\}_{k=1}^n)$ is $\delta_i(x)$-fine, $i = 1, 2$. Set

$$\delta(x) = \min\{\delta_1(x),\, \delta_2(x)\}.$$

Then $\delta(x) > 0$ for every $x \in [a, b]$, so δ is a gauge in the sense of Definition 8.1.3, and any $\delta(x)$-fine tagged partition is simultaneously $\delta_1(x)$-fine and $\delta_2(x)$-fine, because $x_k - x_{k-1} < \delta(c_k) \leq \delta_i(c_k)$.

Theorem 8.1.5 supplies a tagged partition $(P, \{c_k\}_{k=1}^n)$ of $[a, b]$ that is $\delta(x)$-fine – without this the estimate below would be vacuous. For that partition,

$$\begin{aligned}|A_1 - A_2| &\leq |A_1 - R(f, P)| + |R(f, P) - A_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus $|A_1 - A_2| < \epsilon$ for every $\epsilon > 0$, and therefore $A_1 = A_2$ by Theorem 1.2.6.

Problem 8.1.9. Explain why every function that is Riemann-integrable with $\int_a^b f = A$ must also have generalized Riemann integral A.

Solution. Because a constant is a gauge.
Let f be Riemann-integrable on $[a, b]$ with $\int_a^b f = A$, and let $\epsilon > 0$. Riemann-integrable functions are bounded by Definition 7.2.7, so Theorem 8.1.2 applies in the forward direction and produces a constant $\delta > 0$ such that

$$|R(f, P) - A| < \epsilon$$

for every tagged partition $(P, \{c_k\})$ that is δ-fine in the sense of Definition 8.1.1.
Now define $\delta(x) = \delta$ for all $x \in [a, b]$; since $\delta > 0$ this is a gauge (Definition 8.1.3). For a constant gauge the two notions of fineness coincide, since $x_k - x_{k-1} < \delta(c_k)$ for every k (Definition 8.1.4) says exactly that $x_k - x_{k-1} < \delta$ for every k (Definition 8.1.1). So every $\delta(x)$-fine tagged partition obeys the displayed inequality, and as $\epsilon > 0$ was arbitrary, Definition 8.1.6 holds with the value A.

Problem 8.1.10. This exercise completes the proof of Theorem 8.1.8: *Dirichlet's function g is generalized Riemann-integrable on* $[0, 1]$ *with* $\int_0^1 g = 0$. Here

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

$\epsilon > 0$ is given, $\{r_1, r_2, r_3, \ldots\}$ is an enumeration of the rational numbers in $[0, 1]$, and the gauge δ on $[0, 1]$ is defined by

$$\delta(r_j) = \frac{\epsilon}{2^{j+1}} \quad \text{for each } j,$$
$$\delta(x) = 1 \quad \text{for } x \text{ irrational.}$$

Show that if $(P, \{c_k\}_{k=1}^n)$ is a $\delta(x)$-fine tagged partition, then $R(g, P) < \epsilon$.

Solution. Only the rationally tagged subintervals contribute, and each rational r_j contributes less than $2\delta(r_j) = \epsilon/2^j$.
Write $\Delta x_k = x_k - x_{k-1}$. Since $g(c_k) = 0$ whenever c_k is irrational and $g(c_k) = 1$ whenever c_k is rational,

$$\begin{aligned} R(g, P) &= \sum_{k=1}^{n} g(c_k)\, \Delta x_k \\ &= \sum_{k\,:\,c_k \in \mathbf{Q}} \Delta x_k \;\geq\; 0. \end{aligned}$$

If no tag is rational this sum is $0 < \epsilon$ and we are done. Otherwise let

$$J = \{\, j \in \mathbf{N} : r_j = c_k \text{ for some } k \,\},$$

a nonempty set of at most n indices. Each r_j is a tag for at most two values of k, since a point lies in two of the subintervals $[x_{k-1}, x_k]$ only when it is one of the interior partition points $x_1, \ldots, x_{n-1}$; and fineness gives $\Delta x_k < \delta(c_k) = \delta(r_j)$ for each such k. Grouping the sum by the value of the tag,

$$\begin{aligned} R(g, P) &= \sum_{j \in J} \sum_{k\,:\,c_k = r_j} \Delta x_k \;<\; \sum_{j \in J} 2\,\delta(r_j) \\ &= \sum_{j \in J} \frac{2\epsilon}{2^{j+1}} \;=\; \epsilon \sum_{j \in J} \frac{1}{2^j} \;\leq\; \epsilon \sum_{j=1}^{\infty} \frac{1}{2^j} \;=\; \epsilon. \end{aligned}$$

Hence $0 \leq R(g, P) < \epsilon$ for every $\delta(x)$-fine tagged partition, which is precisely Definition 8.1.6 with $A = 0$.

Problem 8.1.11. This exercise is a step in the proof of Theorem 8.1.9 (the Fundamental Theorem of Calculus for the generalized Riemann integral). Let $F : [a, b] \to \mathbf{R}$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$

be a partition of $[a, b]$, so that $a = x_0 < x_1 < \cdots < x_n = b$. Show that

$$F(b) - F(a) \;=\; \sum_{k=1}^{n} \big[F(x_k) - F(x_{k-1})\big].$$

Solution. The sum telescopes.
Induct on n. For $n = 1$ the right side is $F(x_1) - F(x_0) = F(b) - F(a)$. Granting the identity for every partition into n subintervals, take $x_0 < x_1 < \cdots < x_{n+1}$ and split off the last term:

$$\begin{aligned}\sum_{k=1}^{n+1}\big[F(x_k) - F(x_{k-1})\big] &= \big[F(x_n) - F(x_0)\big] \\ &\qquad + F(x_{n+1}) - F(x_n) \\ &= F(x_{n+1}) - F(x_0),\end{aligned}$$

the first equality being the induction hypothesis applied to the partition $\{x_0, \ldots, x_n\}$ of $[x_0, x_n]$. With $x_0 = a$ and $x_{n+1} = b$ this is $F(b) - F(a)$.

Problem 8.1.12. This exercise constructs the gauge used in the proof of Theorem 8.1.9. Assume $F : [a, b] \to \mathbf{R}$ is differentiable at each point of $[a, b]$ and set $f(x) = F'(x)$; let $\epsilon > 0$ be given. For each $c \in [a, b]$, explain why there exists a $\delta(c) > 0$ (a $\delta > 0$ depending on c) such that

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \epsilon$$

for all $0 < |x - c| < \delta(c)$.

Solution. This is the definition of the derivative, read as an ϵ-δ statement.
Fix $c \in [a, b]$. Differentiability of F at c says, by Definition 5.2.1, exactly that

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c} \;=\; F'(c) \;=\; f(c),$$

the limit taken over $x \in [a, b]$ with $x \neq c$; the point c is a limit point of $[a, b]$ (approached from one side when $c = a$ or $c = b$), so Definition 4.2.1 applies. Fed the given ϵ, it returns a $\delta > 0$ with

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \epsilon \quad \text{whenever } 0 < |x - c| < \delta,$$

x ranging over $[a, b]$. Call that number $\delta(c)$; carrying this out at each c defines $\delta : [a, b] \to \mathbf{R}$ with $\delta(c) > 0$ everywhere, a gauge in the sense of Definition 8.1.3.

Problem 8.1.13. This exercise completes the proof of Theorem 8.1.9: *if $F : [a, b] \to \mathbf{R}$ is differentiable at each point of $[a, b]$ and $f(x) = F'(x)$, then f has generalized Riemann integral $\int_a^b f = F(b) - F(a)$.* Let $\epsilon > 0$, let $\delta(c)$ be the gauge built in Exercise 8.1.12, and let $(P, \{c_k\}_{k=1}^{n})$ be a $\delta(c)$-fine tagged partition of $[a, b]$, where $P = \{x_0, x_1, \ldots, x_n\}$.
(a) For a particular $c_k \in [x_{k-1}, x_k]$ of P, show that

$$\big|F(x_k) - F(c_k) - f(c_k)(x_k - c_k)\big| < \epsilon\,(x_k - c_k)$$

and

$$\big|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})\big| < \epsilon\,(c_k - x_{k-1}).$$

(b) Now, argue that

$$\big|F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})\big| < \epsilon\,(x_k - x_{k-1}),$$

and use this fact to complete the proof of the theorem, i.e. to establish the estimate labelled (2) in the text,

$$\big|F(b) - F(a) - R(f, P)\big| < \epsilon.$$

Solution. Multiply the estimate of Exercise 8.1.12 through by the (positive) length of each half of the subinterval, then add the two halves.

Both inequalities in (a) must be read as $\leq$ in the degenerate case where c_k equals the endpoint in question, both sides then being 0; that is all part (b) needs.

(a) The partition is $\delta(c)$-fine, so $x_k - x_{k-1} < \delta(c_k)$.

(i) If $x_k > c_k$, then $0 < |x_k - c_k| \leq x_k - x_{k-1} < \delta(c_k)$, so Exercise 8.1.12 applies with $c = c_k$ and $x = x_k$. Multiplying that inequality by $x_k - c_k > 0$,

$$\begin{aligned}\big|F(x_k) - F(c_k) - f(c_k)(x_k - c_k)\big| &= (x_k - c_k)\left|\frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c_k)\right| \\ &< \epsilon\,(x_k - c_k).\end{aligned}$$

(ii) If $x_k = c_k$, both sides equal 0.

The second inequality is the same computation with $x = x_{k-1}$. When $c_k > x_{k-1}$ we have $0 < |x_{k-1} - c_k| \leq x_k - x_{k-1} < \delta(c_k)$, and multiplying by $c_k - x_{k-1} > 0$ gives

$$\big|F(x_{k-1}) - F(c_k) - f(c_k)(x_{k-1} - c_k)\big| < \epsilon\,(c_k - x_{k-1});$$

the left-hand side is unchanged by negating its argument, so it equals $|F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1})|$. When $c_k = x_{k-1}$ both sides are 0.

(b) Write $\Delta x_k = x_k - x_{k-1}$. Since $(x_k - c_k) + (c_k - x_{k-1}) = \Delta x_k$, the bracketed quantity in (b) is the sum of the two bracketed quantities in (a), so the triangle inequality and part (a) give

$$\begin{aligned}\big|F(x_k) - F(x_{k-1}) - f(c_k)\,\Delta x_k\big| &< \epsilon\,(x_k - c_k) + \epsilon\,(c_k - x_{k-1}) \\ &= \epsilon\,\Delta x_k,\end{aligned}$$

the strictness surviving because $x_{k-1} < x_k$ forces at least one of the two half-lengths to be positive, hence at least one of the two estimates in (a) to be strict while the other reads $0 \leq 0$.

Now sum over k and use Exercise 8.1.11 together with $R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k$:

$$\begin{aligned}\big|F(b) - F(a) - R(f, P)\big| &= \left|\sum_{k=1}^{n}\big[F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k\big]\right| \\ &\leq \sum_{k=1}^{n}\big|F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k\big| \\ &< \sum_{k=1}^{n}\epsilon\,\Delta x_k \;=\; \epsilon\,(b - a).\end{aligned}$$

To land on (2) exactly rather than on $\epsilon(b - a)$, build the gauge of Exercise 8.1.12 from $\epsilon/(b - a)$ in place of ϵ – the intended reading, and no change to the construction, since $\epsilon/(b - a)$ is just another positive number. Every $\delta(c)$-fine tagged partition then satisfies $|F(b) - F(a) - R(f, P)| < \epsilon$, and as $\epsilon > 0$ was arbitrary, Definition 8.1.6 is met with $A = F(b) - F(a)$.

Problem 8.1.14. This exercise completes the proof of Theorem 8.1.10 (Change-of-variable Formula): *Let $g : [a, b] \to \mathbf{R}$ be differentiable at each point of $[a, b]$, and assume F is differentiable on the set*

$g([a,b])$. *If* $f(x) = F'(x)$ *for all* $x \in g([a,b])$, *then*

$$\int_a^b (f \circ g) \cdot g' \;=\; \int_{g(a)}^{g(b)} f.$$

The hypothesis of the theorem guarantees that the function $(F \circ g)(x)$ is differentiable for all $x \in [a,b]$.
(a) Why are we sure that f and $(F \circ g)'$ have generalized Riemann integrals?
(b) Use Theorem 8.1.9 to finish the proof.

Solution. Both are derivatives of functions differentiable on the relevant closed interval, so Theorem 8.1.9 applies to each.
(a) For $(F \circ g)'$: g is differentiable on $[a,b]$, and F is differentiable at every point of $g([a,b])$, so the Chain Rule (Theorem 5.2.5) makes $F \circ g$ differentiable at each point of $[a,b]$, with

$$\begin{aligned}(F \circ g)'(x) &= F'(g(x))\, g'(x) = f(g(x))\, g'(x)\\ &= \big[(f \circ g) \cdot g'\big](x).\end{aligned}$$

Theorem 8.1.9, applied to the function $F \circ g$ on $[a,b]$, therefore says that $(F \circ g)' = (f \circ g) \cdot g'$ has a generalized Riemann integral on $[a,b]$, with

$$\begin{aligned}\int_a^b (f \circ g) \cdot g' &= (F \circ g)(b) - (F \circ g)(a)\\ &= F(g(b)) - F(g(a)).\end{aligned}$$

For f: Theorem 8.1.9 must be applied to F on the closed interval with endpoints $g(a)$ and $g(b)$, so that interval has to lie in the set where F is differentiable. It does: g is continuous on $[a,b]$ (Theorem 5.2.3, since it is differentiable there), so the Intermediate Value Theorem (Theorem 4.5.1) puts every value between $g(a)$ and $g(b)$ in $g([a,b])$, where F is differentiable with $F' = f$ by hypothesis. Hence

$$\int_{g(a)}^{g(b)} f \;=\; F(g(b)) - F(g(a)),$$

read when $g(b) < g(a)$ through the convention of Definition 7.4.3, which reverses the limits and negates both sides at once; when $g(a) = g(b)$ both sides are 0.
(b) The two displays in (a) have the same right-hand side, so

$$\int_a^b (f \circ g) \cdot g' \;=\; F(g(b)) - F(g(a)) \;=\; \int_{g(a)}^{g(b)} f.$$

8.3 Exercises 8.2.1–8.2.7

Problem 8.2.1. Decide which of the following are metrics on $X = \mathbf{R}^2$. For each, we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the plane.
(a) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
(b) $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.
(c) $d(x,y) = |x_1 x_2 + y_1 y_2|$.

Solution. (a) and (b) are metrics; (c) is not.
(a) Write $|u| = \sqrt{u_1^2 + u_2^2}$, so that $d(x,y) = |x - y|$. Conditions (i) and (ii) of Definition 8.2.1 are immediate: $|u| \ge 0$ with $|u| = 0$ exactly when $u_1 = u_2 = 0$, and $|-u| = |u|$. For (iii), the Cauchy–Schwarz inequality $|u_1 v_1 + u_2 v_2| \le |u||v|$ (square both sides; it reduces to $(u_1 v_2 - u_2 v_1)^2 \ge 0$) gives

$$\begin{aligned}|u + v|^2 &= |u|^2 + 2(u_1 v_1 + u_2 v_2) + |v|^2\\ &\le |u|^2 + 2|u||v| + |v|^2 = (|u| + |v|)^2,\end{aligned}$$

so $|u+v| \le |u| + |v|$. Taking $u = x - z$ and $v = z - y$ yields $d(x,y) \le d(x,z) + d(z,y)$.
(b) Again (i) and (ii) are clear, since $\max\{|x_1 - y_1|, |x_2 - y_2|\} = 0$ forces both coordinates to agree. For (iii), fix z and note that for each index $i \in \{1,2\}$,

$$|x_i - y_i| \le |x_i - z_i| + |z_i - y_i| \le d(x,z) + d(z,y).$$

The right-hand side is a bound for both $i = 1$ and $i = 2$, hence for their maximum: $d(x,y) \le d(x,z) + d(z,y)$.
(c) This fails condition (i) in both directions. Taking $x = y = (1,1)$ gives $d(x,x) = |1+1| = 2 \ne 0$, so a point is at positive distance from itself; and taking $x = (1,0)$, $y = (0,1)$ gives $d(x,y) = |0+0| = 0$ for two distinct points.

Problem 8.2.2. Let $C[0,1]$ be the collection of continuous functions on the closed interval $[0,1]$. Decide which of the following are metrics on $C[0,1]$.
(a) $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$.
(b) $d(f,g) = |f(1) - g(1)|$.
(c) $d(f,g) = \int_0^1 |f - g|$.

Solution. (a) and (c) are metrics; (b) is not.
(a) The supremum is finite because $f - g$ is continuous on the compact set $[0,1]$, hence bounded by the Extreme Value Theorem (Theorem 4.4.2). Condition (i): $d(f,g) \ge 0$ always, and $d(f,g) = 0$ forces $|f(x) - g(x)| = 0$ for every x, i.e. $f = g$ as functions. Condition (ii) holds since $|f(x) - g(x)| = |g(x) - f(x)|$ pointwise. For (iii), let $h \in C[0,1]$; for each $x \in [0,1]$,

$$|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le d(f,h) + d(h,g),$$

and taking the supremum over x on the left gives $d(f,g) \le d(f,h) + d(h,g)$.
(b) This fails the *only if* half of condition (i): $f(x) = x$ and $g(x) = x^2$ are distinct elements of $C[0,1]$ with $d(f,g) = |1 - 1| = 0$.
(c) Each $|f - g|$ is continuous, hence integrable by Theorem 7.2.9, so d is well defined and $d(f,g) \ge 0$. Symmetry is immediate, and the triangle inequality follows from the pointwise bound $|f - g| \le |f - h| + |h - g|$ together with monotonicity and additivity of the integral:

$$\int_0^1 |f - g| \le \int_0^1 |f - h| + \int_0^1 |h - g|.$$

The one substantive point is that $d(f,g) = 0$ implies $f = g$. Suppose $|f(x_0) - g(x_0)| = c > 0$ for some $x_0 \in [0,1]$. By continuity of $|f - g|$ there is an interval $I \subseteq [0,1]$ containing x_0, of some length $\delta > 0$, on which $|f - g| > c/2$; since $|f - g| \ge 0$ everywhere,

$$\int_0^1 |f - g| \;\ge\; \int_I |f - g| \;\ge\; \frac{c\delta}{2} \;>\; 0.$$

Hence $d(f,g) = 0$ forces $f = g$.

Problem 8.2.3. The following distance function is called the *discrete metric* and can be defined on any set X. For any $x, y \in X$, let

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

Verify that the discrete metric is actually a metric.

Solution. All three conditions of Definition 8.2.1 are read off from the definition.
(i) ρ takes only the values 0 and 1, so $\rho \ge 0$, and $\rho(x,y) = 0$ holds precisely in the case $x = y$.

(ii) The condition $x \neq y$ is symmetric in x and y, so $\rho(x, y) = \rho(y, x)$.
(iii) Fix $x, y, z \in X$ and split on whether $x = y$.
(1) If $x = y$, then $\rho(x, y) = 0 \leq \rho(x, z) + \rho(z, y)$, since the right side is a sum of nonnegative terms.
(2) If $x \neq y$, then z cannot equal both x and y, so at least one of $\rho(x, z)$, $\rho(z, y)$ equals 1 and

$$\rho(x, y) = 1 \leq \rho(x, z) + \rho(z, y).$$

Problem 8.2.4. Show that a convergent sequence is Cauchy. (That is, if (x_n) is a sequence in a metric space (X, d) converging to $x \in X$ in the sense of Definition 8.2.2, then (x_n) is a Cauchy sequence in the sense of Definition 8.2.3.)

Solution. Let $\epsilon > 0$. Since $x_n \to x$, Definition 8.2.2 supplies an $N \in \mathbf{N}$ with $d(x_n, x) < \epsilon/2$ whenever $n \geq N$. Then for all $m, n \geq N$, conditions (iii) and (ii) of Definition 8.2.1 give

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is precisely the Cauchy condition of Definition 8.2.3.

Problem 8.2.5. (a) Consider $\mathbf{R}^2$ with the discrete metric $\rho(x, y)$ examined in Exercise 8.2.3. What do Cauchy sequences look like in this space? Is $\mathbf{R}^2$ complete with respect to this metric?
(b) Show that $C[0, 1]$ is complete with respect to the metric in Exercise 8.2.2 (a).
(c) Define $C^1[0, 1]$ to be the collection of differentiable functions on $[0, 1]$ whose derivatives are also continuous. Is $C^1[0, 1]$ complete with respect to the metric defined in Exercise 8.2.2 (a)?

Solution. (a) The Cauchy sequences are exactly the eventually constant ones, and $\mathbf{R}^2$ is complete. Indeed, if (x_n) is ρ-Cauchy, apply Definition 8.2.3 with $\epsilon = 1$ to get N with $\rho(x_m, x_n) < 1$ for all $m, n \geq N$; since ρ only takes the values 0 and 1, this says $\rho(x_m, x_n) = 0$, i.e. $x_m = x_n$ for all $m, n \geq N$. So the sequence is constantly equal to some $x = x_N$ from N on. Conversely such a sequence converges to that x, because $\rho(x_n, x) = 0 < \epsilon$ for every $n \geq N$ and every $\epsilon > 0$. Hence every Cauchy sequence converges in $\mathbf{R}^2$, and Definition 8.2.4 is satisfied.
(b) Let (f_n) be Cauchy in $C[0, 1]$ with respect to $d(f, g) = \|f - g\|_\infty$. Given $\epsilon > 0$, choose N with $d(f_m, f_n) < \epsilon$ for $m, n \geq N$; since $|f_m(x) - f_n(x)| \leq d(f_m, f_n)$ for every $x \in [0, 1]$, we get

$$|f_m(x) - f_n(x)| < \epsilon \quad \begin{array}{l} \text{for all } x \in [0, 1] \\ \text{and all } m, n \geq N, \end{array}$$

which is exactly the hypothesis of the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5). Hence there is a function f on $[0, 1]$ with $f_n \to f$ uniformly. Each f_n is continuous, so the Continuous Limit Theorem (Theorem 6.2.6) gives $f \in C[0, 1]$.
It remains to see that uniform convergence *is* convergence in this metric. Given $\epsilon > 0$, uniformity supplies N with $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in [0, 1]$ and $n \geq N$; taking the supremum over x gives $d(f_n, f) \leq \epsilon/2 < \epsilon$ for $n \geq N$. Thus $f_n \to f$ in $C[0, 1]$, and $C[0, 1]$ is complete.
(c) No. Take

$$f_n(x) = \sqrt{\left(x - \tfrac{1}{2}\right)^2 + \tfrac{1}{n}}, \qquad x \in [0, 1].$$

Each f_n lies in $C^1[0, 1]$: the radicand is bounded below by $1/n > 0$, so

$$f_n{}'(x) = \frac{x - \frac{1}{2}}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}}}$$

exists and is continuous on $[0,1]$. Writing $t = x - \frac{1}{2}$ and rationalizing,

$$0 \le \sqrt{t^2 + \tfrac{1}{n}} - |t| = \frac{1/n}{\sqrt{t^2 + \frac{1}{n}} + |t|} \le \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}},$$

so $d(f_n, f) \le 1/\sqrt{n} \to 0$ where $f(x) = |x - \frac{1}{2}|$. Convergence in the metric implies the sequence is Cauchy (Exercise 8.2.4), so (f_n) is a Cauchy sequence in $C^1[0,1]$.
But it has no limit in $C^1[0,1]$. Since $|g(x) - f_n(x)| \le d(g, f_n)$ for each fixed x, convergence in this metric implies pointwise convergence, and pointwise limits are unique; so any $g \in C^1[0,1]$ with $d(f_n, g) \to 0$ would satisfy $g(x) = |x - \frac{1}{2}|$ for all x. That function is not differentiable at $x = 1/2$, since its difference quotients tend to -1 from the left and $+1$ from the right. Hence $g \notin C^1[0,1]$, and $C^1[0,1]$ is not complete.

Problem 8.2.6. Which of these functions from $C[0,1]$ to $\mathbf{R}$ (with the usual metric) are continuous?
(a) $g(f) = \int_0^1 fk$, where k is some fixed function in $C[0,1]$.
(b) $g(f) = f(1/2)$.
(c) $g(f) = f(1/2)$, but this time with respect to the metric on $C[0,1]$ from Exercise 8.2.2 (c).
(Here $C[0,1]$ carries the sup metric $d(f,h) = \|f - h\|_\infty$ of Exercise 8.2.2 (a) in parts (a) and (b), and the metric $d(f,h) = \int_0^1 |f - h|$ in part (c).)

Solution. (a) and (b) are continuous; (c) is not.
(a) The function k is continuous on the compact interval $[0,1]$, so $\|k\|_\infty < \infty$ by the Extreme Value Theorem (Theorem 4.4.2). For $f, h \in C[0,1]$, linearity of the integral and the bound $\left|\int_0^1 u\right| \le \int_0^1 |u|$ give

$$|g(f) - g(h)| = \left|\int_0^1 (f-h)k\right| \le \int_0^1 |f-h|\,|k| \le \int_0^1 \|f-h\|_\infty \|k\|_\infty = \|k\|_\infty\, d(f,h).$$

So given $\epsilon > 0$, Definition 8.2.5 is met at every f with $\delta = \epsilon/(\|k\|_\infty + 1)$.
(b) Even more directly: $|g(f) - g(h)| = |f(1/2) - h(1/2)| \le \|f - h\|_\infty = d(f,h)$, so $\delta = \epsilon$ works at every f.
(c) Evaluation at $1/2$ is not continuous at $f = 0$ for the metric $d(f,h) = \int_0^1 |f - h|$, because a function can have a tall spike of negligible area. For $n \ge 2$ let h_n be the tent

$$h_n(x) = \max\left\{0,\ 1 - n\left|x - \tfrac{1}{2}\right|\right\},$$

which is continuous, equals 1 at $x = 1/2$, and vanishes outside $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] \subseteq [0,1]$. Its graph is a triangle of base $2/n$ and height 1, so

$$d(h_n, 0) = \int_0^1 |h_n| = \frac{1}{2} \cdot \frac{2}{n} \cdot 1 = \frac{1}{n} \longrightarrow 0,$$
$$|g(h_n) - g(0)| = |1 - 0| = 1.$$

Take $\epsilon = 1/2$. For any $\delta > 0$ choose $n > 1/\delta$; then $d(h_n, 0) < \delta$ but $|g(h_n) - g(0)| = 1 \ge \epsilon$, so no δ satisfies Definition 8.2.5 at $f = 0$.

Problem 8.2.7. Describe the ϵ-neighborhoods in $\mathbf{R}^2$ for each of the different metrics described in Exercise 8.2.1. How about for the discrete metric?

Solution. A disk, an axis-parallel square, and either a single point or everything.
(a) For the Euclidean metric $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, the set $V_\epsilon(x) = \{y : d(x,y) < \epsilon\}$ is the open disk of radius ϵ centered at x — the interior of the circle of radius ϵ about x, with the circle itself excluded.
(b) For $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$, the inequality $d(x,y) < \epsilon$ says that *both* $|x_1-y_1| < \epsilon$ and $|x_2-y_2| < \epsilon$, so

$$V_\epsilon(x) = (x_1-\epsilon,\, x_1+\epsilon) \times (x_2-\epsilon,\, x_2+\epsilon),$$

the open square of side length 2ϵ centered at x with sides parallel to the coordinate axes (boundary excluded).
(Part (c) of Exercise 8.2.1 is not a metric, so it has no ϵ-neighborhoods to describe.)
For the discrete metric ρ on any set X, the only possible distances are 0 and 1, so

$$V_\epsilon(x) = \begin{cases} \{x\} & \text{if } 0 < \epsilon \le 1, \\ X & \text{if } \epsilon > 1. \end{cases}$$

In $\mathbf{R}^2$ this means every neighborhood is either the single point x or the whole plane.

8.4 Exercises 8.2.8–8.2.14

Problem 8.2.8. Let (X,d) be a metric space.
(a) Verify that a typical ϵ-neighborhood $V_\epsilon(x)$ is an open set. Is the set

$$C_\epsilon(x) = \{y \in X : d(x,y) \le \epsilon\}$$

a closed set?
(b) Show that a set $E \subseteq X$ is open if and only if its complement is closed.

Solution. (a) Given $y \in V_\epsilon(x)$, the radius $\delta = \epsilon - d(x,y) > 0$ works: if $d(y,z) < \delta$ then

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \delta = \epsilon,$$

so $V_\delta(y) \subseteq V_\epsilon(x)$ and $V_\epsilon(x)$ is open.
Yes, $C_\epsilon(x)$ is closed. Let y be a limit point of $C_\epsilon(x)$. For each $n \in \mathbf{N}$ the neighborhood $V_{1/n}(y)$ meets $C_\epsilon(x)$, say in y_n, and then

$$d(x,y) \le d(x,y_n) + d(y_n,y) < \epsilon + \tfrac{1}{n}$$

for every n, so $d(x,y) \le \epsilon$ and $y \in C_\epsilon(x)$.
(b) ($\Rightarrow$) Let E be open and let y be a limit point of E^c. If y were in E, then $V_\epsilon(y) \subseteq E$ for some $\epsilon > 0$, and this neighborhood would contain no point of E^c at all – contradicting that y is a limit point of E^c. Hence $y \in E^c$, and E^c contains its limit points.
($\Leftarrow$) Let E^c be closed and let $x \in E$. If no ϵ-neighborhood of x were contained in E, then every $V_\epsilon(x)$ would meet E^c, necessarily in a point other than x (as $x \in E$); thus x would be a limit point of E^c and so $x \in E^c$, a contradiction. Hence some $V_\epsilon(x) \subseteq E$, and E is open.

Problem 8.2.9. (a) Show that the set $Y = \{f \in C[0,1] : \|f\|_\infty \le 1\}$ is closed in $C[0,1]$.
(b) Is the set $T = \{f \in C[0,1] : f(0) = 0\}$ open, closed, or neither in $C[0,1]$?

Solution. (a) Y is exactly the set $C_1(0) = \{f : d(f,0) \le 1\}$ centered at the zero function, since $d(f,0) = \|f\|_\infty$, and such sets are closed by Exercise 8.2.8 (a).
(b) Closed, but not open. For closedness, let f be a limit point of T and pick $f_n \in T$ with $d(f,f_n) < 1/n$; then

$$|f(0)| = |f(0) - f_n(0)| \le \|f - f_n\|_\infty < \tfrac{1}{n}$$

for every n, so $f(0) = 0$ and $f \in T$.

For failure of openness, take $f = 0 \in T$ and let $\epsilon > 0$ be arbitrary. The constant function $g \equiv \epsilon/2$ satisfies $d(0, g) = \epsilon/2 < \epsilon$, so $g \in V_\epsilon(0)$, while $g(0) = \epsilon/2 \neq 0$ puts $g \notin T$. Thus no ϵ-neighborhood of 0 lies inside T.

Problem 8.2.10. (a) Supply a definition for bounded subsets of a metric space (X, d).
(b) Show that if K is a compact subset of the metric space (X, d), then K is closed and bounded.
(c) Show that $Y \subseteq C[0, 1]$ from Exercise 8.2.9 (a) is closed and bounded but not compact.

Solution. (a) A set $E \subseteq X$ is bounded if $E = \emptyset$ or there exist $x_0 \in X$ and $M > 0$ with $E \subseteq V_M(x_0)$, that is, $d(x_0, y) < M$ for all $y \in E$.
(b) Closed: let y be a limit point of K and choose $y_n \in K$ with $d(y, y_n) < 1/n$, so that $y_n \to y$. By Definition 8.2.7 some subsequence (y_{n_k}) converges to a limit in K; but every subsequence of a convergent sequence converges to the same limit y, so $y \in K$.
Bounded: assume $K \neq \emptyset$, fix $x_0 \in X$, and suppose K is unbounded. Then for each n there is $y_n \in K$ with $d(x_0, y_n) \geq n$. Compactness gives a subsequence $y_{n_k} \to y \in K$, and then

$$n_k \leq d(x_0, y_{n_k}) \leq d(x_0, y) + d(y, y_{n_k}) \leq d(x_0, y) + 1$$

for all large k, which is absurd.
(c) Y is closed by part (a) of Exercise 8.2.9 and bounded because $Y \subseteq V_2(0)$. It is not compact: the functions $f_n(x) = x^n$ lie in Y since $\|f_n\|_\infty = 1$, yet no subsequence converges in $C[0, 1]$. Indeed, uniform convergence $f_{n_k} \to f$ forces pointwise convergence, and

$$\lim_{k\to\infty} x^{n_k} = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1, \end{cases}$$

so f would have to be this discontinuous function, which is not an element of $C[0, 1]$.

Problem 8.2.11. (a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.
(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Solution. (a) Write L for the set of limit points of E, so that $\overline{E} = E \cup L$ by Definition 8.2.8. Then $\overline{E} = E$ holds precisely when $L \subseteq E$, which is the definition of E being closed.
For the interior, $E^\circ \subseteq E$ always holds by Definition 8.2.8, so $E^\circ = E$ is equivalent to $E \subseteq E^\circ$, i.e. to the statement that every $x \in E$ admits some $V_\epsilon(x) \subseteq E$. That is exactly the definition of E open.
(b) Both inclusions come from unwinding the two definitions at a single point. If $x \in \overline{E}^c$, then $x \notin E$ and x is not a limit point of E, so there is an $\epsilon > 0$ with $V_\epsilon(x) \cap E \subseteq \{x\}$; since $x \notin E$ this intersection is empty, giving

$$x \in E^c \quad \text{and} \quad V_\epsilon(x) \subseteq E^c,$$

i.e. $x \in (E^c)^\circ$. Conversely, if $x \in (E^c)^\circ$ then $x \in E^c$ and $V_\epsilon(x) \subseteq E^c$ for some $\epsilon > 0$; the latter says $V_\epsilon(x) \cap E = \emptyset$, so x is neither in E nor a limit point of E, and $x \in \overline{E}^c$.
For the dual identity, apply what was just proved with E^c in place of E:

$$\overline{E^c}^c = \big((E^c)^c\big)^\circ = E^\circ.$$

Taking complements of both sides gives $\overline{E^c} = (E^\circ)^c$.

Problem 8.2.12. (a) Show

$$\overline{V_\epsilon(x)} \subseteq \{y \in X : d(x, y) \leq \epsilon\},$$

in an arbitrary metric space (X, d).

(b) To keep things from sounding too familiar, find an example of a specific metric space where

$$\overline{V_\epsilon(x)} \neq \{y \in X : d(x,y) \leq \epsilon\}.$$

Solution. (a) $C_\epsilon(x) = \{y \in X : d(x,y) \leq \epsilon\}$ contains $V_\epsilon(x)$ and is closed by Exercise 8.2.8 (a), so every limit point of $V_\epsilon(x)$ is a limit point of $C_\epsilon(x)$ and therefore lies in it:

$$\overline{V_\epsilon(x)} = V_\epsilon(x) \cup \{\text{limit points}\} \subseteq C_\epsilon(x).$$

(b) Take $X = \{a, b\}$ with the discrete metric of Exercise 8.2.3, so $d(a,b) = 1$ and $d(y,y) = 0$, and let $\epsilon = 1$. Then

$$V_1(a) = \{y : d(a,y) < 1\} = \{a\}, \qquad C_1(a) = \{a,b\} = X.$$

The singleton $\{a\}$ has no limit points (the neighborhood $V_{1/2}(b) = \{b\}$ misses it, and likewise at a), so $\overline{V_1(a)} = \{a\} \neq X = C_1(a)$.

Problem 8.2.13. If E is a subset of a metric space (X,d), show that E is nowhere-dense in X if and only if $\overline{E}^c$ is dense in X.

Solution. Apply the second identity of Exercise 8.2.11 (b), namely $(A^\circ)^c = \overline{A^c}$, to the set $A = \overline{E}$:

$$\overline{\overline{E}^c} = \big((\overline{E})^\circ\big)^c.$$

By Definition 8.2.9, $\overline{E}^c$ is dense exactly when the left-hand side equals X, and E is nowhere-dense exactly when $(\overline{E})^\circ = \emptyset$. Since a subset of X has complement X if and only if it is empty, the displayed equality turns each of these conditions into the other:

$$\overline{\overline{E}^c} = X \iff \big((\overline{E})^\circ\big)^c = X \iff (\overline{E})^\circ = \emptyset.$$

Problem 8.2.14. This exercise completes the proof of Theorem 8.2.10: if (X,d) is a complete metric space and $\{O_n\}$ is a countable collection of dense, open subsets of X, then $\bigcap_{n=1}^\infty O_n$ is not empty. The proof so far: pick $x_1 \in O_1$; because O_1 is open, there exists $\epsilon_1 > 0$ such that $V_{\epsilon_1}(x_1) \subseteq O_1$.
(a) Give the details for why we know there exists a point $x_2 \in V_{\epsilon_1}(x_1) \cap O_2$ and an $\epsilon_2 > 0$ satisfying $\epsilon_2 < \epsilon_1/2$ with $V_{\epsilon_2}(x_2)$ contained in O_2 and

$$\overline{V_{\epsilon_2}(x_2)} \subseteq V_{\epsilon_1}(x_1).$$

(b) Proceed along this line and use the completeness of (X,d) to produce a single point $x \in O_n$ for every $n \in \mathbf{N}$.

Solution. (a) Density of O_2 means $\overline{O_2} = X$, and this forces O_2 to meet every nonempty open set. Indeed, if $V_{\epsilon_1}(x_1) \cap O_2 = \emptyset$, then any $y \in V_{\epsilon_1}(x_1)$ would have some $V_\delta(y) \subseteq V_{\epsilon_1}(x_1)$ (openness, Exercise 8.2.8 (a)) disjoint from O_2, so y would be neither a point nor a limit point of O_2, contradicting $y \in X = \overline{O_2}$. Choose $x_2 \in V_{\epsilon_1}(x_1) \cap O_2$. That set is open – x_2 has a neighborhood inside $V_{\epsilon_1}(x_1)$ and one inside the open set O_2, and the smaller radius $\delta > 0$ gives

$$V_\delta(x_2) \subseteq V_{\epsilon_1}(x_1) \cap O_2.$$

Now set $\epsilon_2 = \min\{\delta/2,\ \epsilon_1/4\} > 0$. Then $\epsilon_2 < \epsilon_1/2$, and $V_{\epsilon_2}(x_2) \subseteq V_\delta(x_2) \subseteq O_2$, while Exercise 8.2.12 (a) with $\epsilon_2 < \delta$ gives

$$\overline{V_{\epsilon_2}(x_2)} \subseteq \{y : d(x_2,y) \leq \epsilon_2\} \subseteq V_\delta(x_2) \subseteq V_{\epsilon_1}(x_1).$$

(b) Repeating (a) verbatim with (x_n, ϵ_n) and the dense open set O_{n+1} in place of (x_1, ϵ_1) and O_2

produces inductively points $x_n \in X$ and radii $\epsilon_n > 0$ with

$$\begin{aligned} V_{\epsilon_n}(x_n) &\subseteq O_n, \qquad \epsilon_{n+1} < \epsilon_n/2, \\ \overline{V_{\epsilon_{n+1}}(x_{n+1})} &\subseteq V_{\epsilon_n}(x_n). \end{aligned}$$

The neighborhoods are nested, so $x_m \in V_{\epsilon_n}(x_n)$ whenever $m > n$, and iterating $\epsilon_{n+1} < \epsilon_n/2$ gives $\epsilon_n \leq \epsilon_1/2^{n-1}$. Hence

$$d(x_n, x_m) < \epsilon_n \leq \frac{\epsilon_1}{2^{n-1}} \qquad (m > n),$$

so (x_n) is Cauchy (Definition 8.2.3) and converges to some $x \in X$ by completeness (Definition 8.2.4). Fix $n \in \mathbf{N}$. Every term x_m with $m \geq n+1$ lies in $V_{\epsilon_{n+1}}(x_{n+1})$, so its limit x is either a point of that set or a limit point of it; either way

$$x \in \overline{V_{\epsilon_{n+1}}(x_{n+1})} \subseteq V_{\epsilon_n}(x_n) \subseteq O_n,$$

and x lies in $\bigcap_{n=1}^{\infty} O_n$.

8.5 Exercises 8.2.15–8.3.3

Problem 8.2.15. Complete the proof of the theorem.
(Theorem 8.2.11, the Baire Category Theorem: a complete metric space is not the union of a countable collection of nowhere-dense sets. The preceding Theorem 8.2.10 states that if (X, d) is a complete metric space and $\{O_n\}$ is a countable collection of dense, open subsets of X, then $\bigcap_{n=1}^{\infty} O_n$ is not empty.)

Solution. Take complements and quote Theorem 8.2.10.
Suppose, for contradiction, that the complete metric space (X, d) satisfies

$$X = \bigcup_{n=1}^{\infty} E_n,$$

with every E_n nowhere-dense in X. Set $O_n = \left(\overline{E_n}\right)^c$.
Each O_n is open, being the complement of the closed set $\overline{E_n}$, and each O_n is dense in X by Exercise 8.2.13 (a set E is nowhere-dense exactly when $\overline{E}^c$ is dense). So $\{O_n\}$ is a countable collection of dense open subsets of the complete space X — precisely the hypothesis of Theorem 8.2.10, whose proof also needs $X \neq \emptyset$ in order to pick $x_1 \in O_1$. Hence $\bigcap_{n=1}^{\infty} O_n \neq \emptyset$.
But De Morgan gives

$$\begin{aligned} \bigcap_{n=1}^{\infty} O_n = \bigcap_{n=1}^{\infty} \left(\overline{E_n}\right)^c &= \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right)^c \\ &\subseteq \left(\bigcup_{n=1}^{\infty} E_n\right)^c = X^c = \emptyset, \end{aligned}$$

the inclusion because $E_n \subseteq \overline{E_n}$ reverses under complementation. This contradiction proves the theorem.
□

Problem 8.2.16. Show that if $f \in C[0,1]$ is differentiable at a point $x \in [0,1]$, then $f \in A_{m,n}$ for some pair $m, n \in \mathbf{N}$.

Here, for each pair of natural numbers m, n,

$$A_{m,n} = \Big\{ f \in C[0,1] : \text{there exists } x \in [0,1] \text{ where } \left| \frac{f(x) - f(t)}{x - t} \right| \le n \text{ whenever } 0 < |x - t| < \tfrac{1}{m} \Big\}.$$

Solution. Take $1/m$ smaller than the δ supplied by the derivative and $n \ge |f'(x)| + 1$.
Because $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ exists, the definition of a functional limit with $\varepsilon = 1$ furnishes a $\delta > 0$ such that

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| \le 1 \qquad \text{for all } t \in [0,1] \text{ with } 0 < |x - t| < \delta,$$

the difference quotient being unchanged by negating numerator and denominator. The triangle inequality then gives

$$\left| \frac{f(x) - f(t)}{x - t} \right| \le |f'(x)| + 1 \qquad \text{whenever } 0 < |x - t| < \delta.$$

By the Archimedean Property (Theorem 1.4.2) choose $m \in \mathbf{N}$ with $1/m < \delta$ and $n \in \mathbf{N}$ with $n \ge |f'(x)| + 1$. Since $0 < |x - t| < 1/m$ forces $0 < |x - t| < \delta$, the point x witnesses

$$\left| \frac{f(x) - f(t)}{x - t} \right| \le n \qquad \text{whenever } 0 < |x - t| < \frac{1}{m},$$

so $f \in A_{m,n}$.

Problem 8.2.17. Fix m and n, let (f_k) be a sequence in $A_{m,n}$ with $f_k \to f$ in $C[0,1]$, and for each k let $x_k \in [0,1]$ be a point with

$$\left| \frac{f_k(x_k) - f_k(t)}{x_k - t} \right| \le n \qquad \text{for all } 0 < |x_k - t| < 1/m.$$

(a) The sequence (x_k) does not necessarily converge, but explain why there exists a subsequence (x_{k_l}) that is convergent. Let $x = \lim(x_{k_l})$.
(b) Prove that $f_{k_l}(x_{k_l}) \to f(x)$.
(c) Now finish the proof that $A_{m,n}$ is closed.

Solution. (a) Bolzano–Weierstrass. The sequence (x_k) lies in $[0,1]$, hence is bounded, so Theorem 2.5.5 supplies a convergent subsequence (x_{k_l}). Its limit x lies in $[0,1]$ because $[0,1]$ is closed.
(b) The two errors split:

$$\begin{aligned} |f_{k_l}(x_{k_l}) - f(x)| &\le |f_{k_l}(x_{k_l}) - f(x_{k_l})| + |f(x_{k_l}) - f(x)| \\ &\le \|f_{k_l} - f\|_\infty + |f(x_{k_l}) - f(x)|\,. \end{aligned}$$

The first term tends to 0 because $f_k \to f$ in the metric of $C[0,1]$, which is the supremum metric; the second tends to 0 because f is continuous at x and $x_{k_l} \to x$. The same estimate, verbatim, shows more generally that

$$t_l \to t \text{ in } [0,1] \quad \Longrightarrow \quad f_{k_l}(t_l) \to f(t). \tag{$*$}$$

(c) We show x witnesses $f \in A_{m,n}$, i.e. that

$$|f(x) - f(t)| \le n\,|x - t| \qquad \text{for every } t \in [0,1] \text{ with } |x - t| < \tfrac{1}{m};$$

the case $t = x$ is trivial, and for $t \ne x$ this is the required bound on the difference quotient.

(i) $t \in (0,1)$. Put $t_l = t + (x_{k_l} - x)$. Since $x_{k_l} \to x$ and t is interior, $t_l \in [0,1]$ for all large l, and

$$0 < |x_{k_l} - t_l| = |x - t| < \tfrac{1}{m}$$

for every l. The defining property of x_{k_l} therefore applies to the point t_l:

$$|f_{k_l}(x_{k_l}) - f_{k_l}(t_l)| \le n\,|x_{k_l} - t_l| = n\,|x - t|.$$

Letting $l \to \infty$ and using (b) on the left term and $(*)$ with $t_l \to t$ on the right term gives $|f(x) - f(t)| \le n|x - t|$.
(ii) $t = 0$ (the case $t = 1$ is identical). Here $0 < x < 1/m$. Choose $s_j \in (0,x)$ with $s_j \to 0$; then $|x - s_j| = x - s_j < 1/m$, so case (i) yields $|f(x) - f(s_j)| \le n|x - s_j|$. Continuity of f at 0 and $s_j \to 0$ give $|f(x) - f(0)| \le n\,|x - 0|$.
Hence $f \in A_{m,n}$, and $A_{m,n}$ is closed.

Problem 8.2.18. A continuous function is called *polygonal* if its graph consists of a finite number of line segments. (Throughout, m, n are fixed, $f \in \overline{A_{m,n}} = A_{m,n}$, $\epsilon > 0$, $V_\epsilon(f)$ is the ϵ-neighborhood of f in $C[0,1]$, and p in part (b) is the function produced in part (a).)
(a) Show that there exists a polygonal function $p \in C[0,1]$ satisfying $\|f - p\|_\infty < \epsilon/2$.
(b) Show that if h is any function in $C[0,1]$ that is bounded by 1, then the function

$$g(x) = p(x) + \frac{\epsilon}{2} h(x)$$

satisfies $g \in V_\epsilon(f)$.
(c) Construct a polygonal function $h(x)$ in $C[0,1]$ that is bounded by 1 and leads to the conclusion $g \notin A_{m,n}$, where g is defined as in (b). Explain how this completes the argument for Theorem 8.2.12.

Solution. (a) Take p to be the piecewise-linear interpolant of f on a fine enough uniform partition. Since f is continuous on the compact set $[0,1]$, it is uniformly continuous there (Theorem 4.4.7): there is a $\delta > 0$ with $|f(s) - f(u)| < \epsilon/4$ whenever $|s - u| < \delta$. Pick $N \in \mathbf{N}$ with $1/N < \delta$, set $x_j = j/N$ for $0 \le j \le N$, and let p be the function that agrees with f at each x_j and is linear on each $[x_{j-1}, x_j]$. This p is polygonal.
For $x \in [x_{j-1}, x_j]$ write $p(x) = \lambda f(x_{j-1}) + (1-\lambda) f(x_j)$ with $\lambda \in [0,1]$. Then

$$\begin{aligned} |p(x) - f(x)| &= \big|\lambda\big(f(x_{j-1}) - f(x)\big) + (1-\lambda)\big(f(x_j) - f(x)\big)\big| \\ &\le \max\{|f(x_{j-1}) - f(x)|,\ |f(x_j) - f(x)|\} \\ &< \frac{\epsilon}{4}, \end{aligned}$$

since $|x - x_{j-1}|$ and $|x - x_j|$ are both at most $1/N < \delta$. Hence $\|f - p\|_\infty \le \epsilon/4 < \epsilon/2$.
(b) $\|f - g\|_\infty \le \|f - p\|_\infty + \frac{\epsilon}{2}\|h\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, so $g \in V_\epsilon(f)$.
(c) Take h to be a sawtooth of amplitude 1 whose teeth are steep enough to swamp the slopes of p. Let $M \ge 0$ be the largest absolute slope among the finitely many segments of p, and use the Archimedean Property to choose $K \in \mathbf{N}$ with

$$\epsilon K > n + M.$$

Define h to be the polygonal function with $h(j/K) = (-1)^j$ for $0 \le j \le K$ and linear in between. Then $|h| \le 1$, and on each interval $[(j-1)/K, j/K]$ the slope of h is $\pm 2K$.
Put $g = p + \frac{\epsilon}{2} h$, which lies in $V_\epsilon(f)$ by (b). The function g is polygonal, its breakpoints being contained in the finite set

$$P \cup \{j/K : 0 \le j \le K\},$$

where P is the set of breakpoints of p. On any interval $[a,b]$ between consecutive points of this set, both p and h are linear, so g is linear there with slope of magnitude

$$\big|\, p' \pm \epsilon K \,\big| \ \ge\ \epsilon K - M \ >\ n,$$

since $|p'| \le M$ on that interval.
Now let $x \in [0,1]$ be arbitrary. It belongs to some such interval $[a,b]$ with $a < b$; choose $t \in [a,b]$ with

$$0 < |x-t| < \min\left\{\tfrac{1}{m},\ b-a\right\},$$

which is possible because $b-a>0$: take t to the right of x if $x<b$, and to the left otherwise. Since x and t lie on the same linear piece,

$$\left|\frac{g(x)-g(t)}{x-t}\right| \ \ge\ \epsilon K - M \ >\ n.$$

So no $x \in [0,1]$ satisfies the defining condition of $A_{m,n}$, and $g \notin A_{m,n}$.
Since $f \in \overline{A_{m,n}} = A_{m,n}$ (the equality by 8.2.17) and $\epsilon > 0$ were both arbitrary, no $V_\epsilon(f)$ is contained in $\overline{A_{m,n}}$; so $\overline{A_{m,n}}$ has empty interior and $A_{m,n}$ is nowhere-dense by Definition 8.2.9.
By 8.2.16, $D \subseteq \bigcup_{m,n\in\mathbf{N}} A_{m,n}$, so

$$D \ =\ \bigcup_{m,n\in\mathbf{N}} (D \cap A_{m,n}),$$

a countable union in which each $D \cap A_{m,n}$ is a subset of the nowhere-dense set $A_{m,n}$ and so is itself nowhere-dense. Therefore D is of first category in $C[0,1]$, proving Theorem 8.2.12. □

Problem 8.3.1. Supply the details to show that when $x = \pi/2$ the product formula in (2),

$$\sin(x) = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots,$$

is equivalent to

$$(3) \qquad \frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right) \cdots \left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right),$$

where the infinite product in (2) is interpreted to be a limit of partial products. (Although it is not necessary for what follows, it might be useful to review the treatment of infinite products in Exercises 2.4.10 and 2.7.10.)

Solution. Pairing the two factors carrying the same n makes the even partial products of (2) at $x = \pi/2$ the reciprocals of the partial products in (3).
Let Q_M be the M-th partial product of the factors following the leading x in (2). At $x = \pi/2$ those factors are $1 - \frac{1}{2n}$ and $1 + \frac{1}{2n}$ for $n = 1, 2, 3, \ldots$, all nonzero, so

$$Q_{2N} = \prod_{n=1}^{N}\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right) = \prod_{n=1}^{N}\frac{(2n-1)(2n+1)}{2n\cdot 2n} = \frac{1}{P_N},$$
$$P_N := \prod_{n=1}^{N}\frac{2n\cdot 2n}{(2n-1)(2n+1)}.$$

($\Rightarrow$) If (2) holds at $x = \pi/2$, then (Q_M) converges, say to L, and $1 = \sin(\pi/2) = \frac{\pi}{2}L$; hence $L = 2/\pi \neq 0$. The subsequence $Q_{2N} = 1/P_N$ also tends to L, and since $L \neq 0$,

$$P_N = \frac{1}{Q_{2N}} \longrightarrow \frac{1}{2/\pi} = \frac{\pi}{2},$$

by the Algebraic Limit Theorem (Theorem 2.3.3), which is (3).
($\Leftarrow$) If $P_N \to \pi/2$ then $Q_{2N} = 1/P_N \to 2/\pi$, while

$$Q_{2N+1} = Q_{2N}\left(1-\frac{1}{2(N+1)}\right) \longrightarrow \frac{2}{\pi},$$

so $(Q_M) \to 2/\pi$, its even- and odd-indexed subsequences exhausting it with a common limit. Multiplying by the leading factor $\pi/2$ returns $\frac{\pi}{2} \cdot \frac{2}{\pi} = 1 = \sin(\pi/2)$, i.e. (2) at $x = \pi/2$.

Problem 8.3.2. Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt.$$

Solution. Apply the Fundamental Theorem of Calculus to hk. The product hk is differentiable on $[a, b]$ with

$$(hk)'(t) = h(t)k'(t) + h'(t)k(t),$$

and this is continuous on $[a, b]$, hence integrable there (Theorem 7.2.9). Part (i) of the Fundamental Theorem of Calculus (Theorem 7.5.1) applies to hk with the integrable derivative $(hk)'$ and yields

$$\int_a^b \big(h(t)k'(t) + h'(t)k(t)\big)dt = h(b)k(b) - h(a)k(a).$$

Both hk' and $h'k$ are continuous, hence integrable, so the left side splits by linearity of the integral (Theorem 7.4.2(i)):

$$\int_a^b h(t)k'(t)\,dt + \int_a^b h'(t)k(t)\,dt = h(b)k(b) - h(a)k(a).$$

Subtracting $\int_a^b h'k$ from both sides gives the formula.

Problem 8.3.3. Throughout, $b_n = \int_0^{\pi/2} \sin^n(x)\,dx$ for $n = 0, 1, 2, \ldots$, so that $b_0 = \pi/2$ and $b_1 = 1$.
(a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation

$$b_n = \frac{n-1}{n}\, b_{n-2} \qquad \text{for all } n \geq 2.$$

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).
(c) Write a general expression for b_{2n} and b_{2n+1}.

Solution. (a) Take $h(x) = \sin^{n-1}(x)$ and $k(x) = -\cos(x)$ in Exercise 8.3.2 (both have continuous derivatives on $[0, \pi/2]$ for $n \geq 2$), so $k'(x) = \sin(x)$ and

$$\begin{aligned} b_n &= \int_0^{\pi/2} \sin^{n-1}(x)\sin(x)\,dx \\ &= \Big[-\sin^{n-1}(x)\cos(x)\Big]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\cos^2(x)\,dx. \end{aligned}$$

The boundary term vanishes: $\cos(\pi/2) = 0$ and $\sin^{n-1}(0) = 0$ because $n - 1 \geq 1$. Substituting $\cos^2 = 1 - \sin^2$ gives

$$b_n = (n-1)\int_0^{\pi/2} \big(\sin^{n-2}(x) - \sin^n(x)\big)dx = (n-1)\big(b_{n-2} - b_n\big),$$

so $n\,b_n = (n-1)b_{n-2}$, which is the stated relation.
(b) From $b_0 = \pi/2$ and $b_1 = 1$:

$$b_0 = \frac{\pi}{2}, \qquad b_2 = \frac{1}{2}b_0 = \frac{\pi}{4}, \qquad b_4 = \frac{3}{4}b_2 = \frac{3\pi}{16};$$

$$b_1 = 1, \qquad b_3 = \frac{2}{3} b_1 = \frac{2}{3}, \qquad b_5 = \frac{4}{5} b_3 = \frac{8}{15}.$$

(c) Iterating the recurrence down to b_0 and to b_1 respectively,

$$b_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{\pi}{2},$$

$$b_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 1 = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)},$$

both for $n \geq 0$ with empty products read as 1; the induction step is one application of (a). (Check!)

8.6 Exercises 8.3.4–8.3.10

Problem 8.3.4. With $b_n = \int_0^{\pi/2} \sin^n(x)\,dx$, and using that $0 \leq \sin^{n+1}(x) \leq \sin^n(x)$ on $[0, \pi/2]$ makes (b_n) decreasing, show

$$\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Wallis's product formula in (3).

Solution. Squeeze the ratio between 1 and $\frac{2n+1}{2n}$. Each b_n is strictly positive (the integrand is positive on $(0, \pi/2)$ and continuous), and (b_n) is decreasing, so $b_{2n+1} \leq b_{2n} \leq b_{2n-1}$. The recurrence of Exercise 8.3.3(a) gives $b_{2n+1} = \frac{2n}{2n+1} b_{2n-1}$, i.e. $b_{2n-1} = \frac{2n+1}{2n} b_{2n+1}$, whence

$$1 \leq \frac{b_{2n}}{b_{2n+1}} \leq \frac{b_{2n-1}}{b_{2n+1}} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}.$$

Since $1 + \frac{1}{2n} \to 1$, the Squeeze Theorem (Exercise 2.3.3) gives $b_{2n}/b_{2n+1} \to 1$.
Now feed in the closed forms of Exercise 8.3.3(c):

$$\begin{aligned} \frac{b_{2n}}{b_{2n+1}} &= \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)} \\ &= \frac{\pi}{2} \cdot \frac{\big[1 \cdot 3 \cdots (2n-1)\big]\big[3 \cdot 5 \cdots (2n+1)\big]}{\big[2 \cdot 4 \cdots (2n)\big]^2} = \frac{\pi}{2} \cdot \frac{1}{P_n}, \end{aligned}$$

where, regrouping the product factor by factor,

$$P_n = \frac{\big[2 \cdot 4 \cdots (2n)\big]^2}{\big[1 \cdot 3 \cdots (2n-1)\big]\big[3 \cdot 5 \cdots (2n+1)\big]} = \prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)}$$

is exactly the n-th partial product in (3). Therefore

$$P_n = \frac{\pi}{2} \cdot \frac{b_{2n+1}}{b_{2n}} \longrightarrow \frac{\pi}{2} \cdot 1 = \frac{\pi}{2},$$

by the Algebraic Limit Theorem (Theorem 2.3.3), and (3) is proved.

Problem 8.3.5. Derive the following alternative form of Wallis's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

(Standard notation for equation (3): $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!$ and $1 \cdot 3 \cdot 5 \cdots (2n+1) = \dfrac{(2n+1)!}{2 \cdot 4 \cdot 6 \cdots (2n)} =$

$\dfrac{(2n+1)!}{2^n n!}$.)

Solution. Rewrite the partial product P_n of (3) in factorials and divide by n. Put

$$A_n = 2 \cdot 4 \cdots (2n) = 2^n n!, \qquad B_n = 1 \cdot 3 \cdots (2n-1) = \frac{(2n)!}{2^n n!},$$

the second identity coming from $(2n)! = A_n B_n$. Since $3 \cdot 5 \cdots (2n+1) = B_n(2n+1)$, the computation in Exercise 8.3.4 gives

$$P_n = \frac{A_n^2}{B_n \cdot B_n(2n+1)} = \frac{t_n^2}{2n+1}, \qquad t_n := \frac{A_n}{B_n} = \frac{2^{2n}(n!)^2}{(2n)!}.$$

By (3), $P_n \to \pi/2$, so by the Algebraic Limit Theorem (Theorem 2.3.3),

$$\frac{t_n^2}{n} = \frac{2n+1}{n} \cdot P_n \longrightarrow 2 \cdot \frac{\pi}{2} = \pi.$$

Thus $\left(t_n/\sqrt{n}\right)^2 \to \pi$ with $t_n/\sqrt{n} > 0$, and since $s \mapsto \sqrt{s}$ is continuous at π, Theorem 4.3.2 (sequential characterization of continuity) yields

$$\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} = \frac{t_n}{\sqrt{n}} \longrightarrow \sqrt{\pi}.$$

Problem 8.3.6. Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution. The n-th derivative of $f(x) = (1-x)^{-1/2}$ on $(-1,1)$ is

$$f^{(n)}(x) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} (1-x)^{-(2n+1)/2},$$

with the empty product read as 1 when $n = 0$. Induction: the case $n = 0$ is the definition of f, and differentiating the displayed formula by the chain rule (the inner derivative of $1-x$ contributing -1, which cancels the -1 in the exponent $-(2n+1)/2$) gives

$$f^{(n+1)}(x) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \cdot \frac{2n+1}{2} (1-x)^{-(2n+3)/2},$$

which is the formula with $n+1$ in place of n.
By Taylor's Formula (Theorem 6.6.2) the Taylor coefficients of f about 0 are $c_n = f^{(n)}(0)/n!$, so

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \, n!} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n},$$

using $2 \cdot 4 \cdots (2n) = 2^n n!$; and $c_0 = 1$. Multiplying numerator and denominator by $2 \cdot 4 \cdots (2n) = 2^n n!$ and using $\big[1 \cdot 3 \cdots (2n-1)\big]\big[2 \cdot 4 \cdots (2n)\big] = (2n)!$ turns this into

$$c_n = \frac{(2n)!}{\left(2^n n!\right)^2} = \frac{(2n)!}{2^{2n}(n!)^2}.$$

(Convergence of this series back to f on $(-1,1)$ is deferred to Exercises 8.3.8-8.3.10.)

Problem 8.3.7. With $c_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$ as in Exercise 8.3.6, so that Exercise 8.3.5 can be rephrased as

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{1}{c_n\sqrt{n}},$$

show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Solution. Both statements come from $c_n\sqrt{n} \to 1/\sqrt{\pi}$. Indeed $c_n > 0$ and $1/(c_n\sqrt{n}) \to \sqrt{\pi} \neq 0$, so by the Algebraic Limit Theorem (Theorem 2.3.3),

$$c_n\sqrt{n} \longrightarrow \frac{1}{\sqrt{\pi}}.$$

(i) Convergence to 0: writing $c_n = (c_n\sqrt{n})/\sqrt{n}$, the numerator is a convergent, hence bounded, sequence, say $c_n\sqrt{n} \le M$ for all $n \ge 1$, so

$$0 < c_n \le \frac{M}{\sqrt{n}} \longrightarrow 0,$$

and $\lim c_n = 0$ by the Squeeze Theorem (Exercise 2.3.3).
(ii) Divergence of $\sum c_n$: take $\epsilon = \frac{1}{2\sqrt{\pi}}$ in the definition of the limit above to get an N with

$$c_n\sqrt{n} > \frac{1}{\sqrt{\pi}} - \frac{1}{2\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}}, \qquad \text{i.e.} \qquad c_n > \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}},$$

for all $n \ge N$. The series $\sum_{n\ge 1} n^{-1/2}$ diverges (a p-series with $p = 1/2 \le 1$, Corollary 2.4.7), so $\sum_{n\ge N} \frac{1}{2\sqrt{\pi}} n^{-1/2}$ diverges as well, and the Comparison Test (Theorem 2.7.4) forces $\sum_{n\ge N} c_n$, hence $\sum_{n=0}^{\infty} c_n$, to diverge.

Problem 8.3.8. Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem (Theorem 6.6.3), show that equation (4),

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n,$$
$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n},$$

is valid for all $|x| < 1/2$. What goes wrong when we try to use this method to prove (4) for $x \in (1/2, 1)$?

Solution. The Lagrange remainder for $f(x) = (1-x)^{-1/2}$ is

$$E_N(x) = c_{N+1}\left(\frac{x}{1-c}\right)^{N+1} \frac{1}{\sqrt{1-c}}, \qquad c \text{ between 0 and } x,$$

and this tends to 0 as soon as $|x|/(1-c)$ stays below 1, which the hypothesis $|x| < 1/2$ guarantees. To get the displayed formula, differentiate f repeatedly:

$$f^{(k)}(t) = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k}(1-t)^{-(2k+1)/2} = k!\,c_k\,(1-t)^{-(2k+1)/2},$$

since $c_k = \big(1\cdot 3\cdots(2k-1)\big)/\big(2^k k!\big)$. (Check by induction: each differentiation multiplies by $(2k+1)/2$ and raises the exponent by one.) Because f is infinitely differentiable on $(-R, R)$ for any $R < 1$, Theorem 6.6.3 applies for each N and supplies a point c strictly between 0 and x with

$$\begin{aligned} E_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}\, x^{N+1} \\ &= c_{N+1}\, \frac{x^{N+1}}{(1-c)^{N+1}} \cdot \frac{1}{\sqrt{1-c}}. \end{aligned}$$

Note $0 < c_{N+1} \le c_0 = 1$, since $c_{k+1}/c_k = (2k+1)/(2k+2) < 1$.
(i) $-1/2 < x \le 0$. Then $x \le c \le 0$, so $1-c \ge 1$, giving $|x|/(1-c) \le |x|$ and $(1-c)^{-1/2} \le 1$. Hence

$$|E_N(x)| \le |x|^{N+1} \longrightarrow 0,$$

because $|x| < 1$.
(ii) $0 \le x < 1/2$. Then $0 \le c \le x$, so $1-c \ge 1-x > 0$ and

$$|E_N(x)| \;\le\; \frac{1}{\sqrt{1-x}} \left(\frac{x}{1-x}\right)^{N+1}.$$

The hypothesis $x < 1/2$ is precisely the statement $x < 1-x$, i.e. $r := x/(1-x) < 1$, so $r^{N+1} \to 0$ and the prefactor $(1-x)^{-1/2}$ is a constant. Hence $E_N(x) \to 0$.
Both cases give $E_N(x) \to 0$, which is (4) for $|x| < 1/2$.
What goes wrong for $x \in (1/2, 1)$: Theorem 6.6.3 locates c only by $0 < c < x$, and c may shift with N. The sole bound available uniformly in N is the worst case $c \to x$,

$$\left(\frac{x}{1-c}\right)^{N+1} \le \left(\frac{x}{1-x}\right)^{N+1},$$

and $x > 1/2$ makes $x/(1-x) > 1$, so this majorant diverges instead of shrinking. Nothing in the theorem prevents c from sitting near x for every N, so the estimate is vacuous there.

Problem 8.3.9. This exercise completes the proof of Theorem 8.3.1 (Integral Remainder Theorem): if f is differentiable $N+1$ times on $(-R, R)$ with $f^{(N+1)}$ continuous, $a_n = f^{(n)}(0)/n!$, and $S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$, then for all $x \in (-R, R)$ the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{1}{N!} \int_0^x f^{(N+1)}(t)(x-t)^N \, dt.$$

The case $x = 0$ is immediate, so fix $x \ne 0$ in $(-R, R)$; treat x as a constant and (to avoid technical distractions) assume $x > 0$.
(a) Show

$$f(x) = f(0) + \int_0^x f'(t)\, dt.$$

(b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\, dt.$$

(c) Continue in this fashion to complete the proof of the theorem.

Solution. (a) On $[0, x]$ the function f' is continuous (for $N \ge 1$ it is differentiable, hence continuous by Theorem 5.2.3; for $N = 0$ it is $f^{(N+1)}$, continuous by hypothesis), so it is integrable by Theorem 7.2.9. Since f is an antiderivative of f' there, part (i) of the Fundamental Theorem of Calculus (Theorem 7.5.1) gives

$$\int_0^x f'(t)\, dt = f(x) - f(0).$$

(b) Apply the integration-by-parts formula of Exercise 8.3.2,

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt,$$

on $[0,x]$ with $h(t) = f'(t)$ and $k(t) = t - x$. Both have continuous derivatives on $[0,x]$ ($h' = f''$ is continuous, k is a polynomial), so the hypotheses hold, and $k'(t) = 1$, $k(x) = 0$, $k(0) = -x$:

$$\begin{aligned}\int_0^x f'(t)\,dt &= f'(x)\cdot 0 - f'(0)(-x) - \int_0^x f''(t)(t-x)\,dt\\ &= f'(0)x + \int_0^x f''(t)(x-t)\,dt.\end{aligned}$$

Substituting into (a) gives the stated identity.
(c) Claim: for every n with $0 \le n \le N$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

The case $n = 0$ is (a) and the case $n = 1$ is (b). Assume it for some $n < N$ and apply Exercise 8.3.2 to the integral with

$$h(t) = f^{(n+1)}(t), \qquad k(t) = -\frac{(x-t)^{n+1}}{n+1}, \qquad k'(t) = (x-t)^n.$$

Here $h' = f^{(n+2)}$ exists and is continuous on $[0,x]$ because $n+2 \le N+1$, so the hypotheses hold. Since $k(x) = 0$ and $k(0) = -x^{n+1}/(n+1)$,

$$\begin{aligned}\int_0^x f^{(n+1)}(t)(x-t)^n dt = &\frac{f^{(n+1)}(0)}{n+1}x^{n+1}\\ &+ \frac{1}{n+1}\int_0^x f^{(n+2)}(t)(x-t)^{n+1}dt.\end{aligned}$$

Dividing by $n!$ turns the first term into $\dfrac{f^{(n+1)}(0)}{(n+1)!}x^{n+1}$ and the second into $\dfrac{1}{(n+1)!}\displaystyle\int_0^x f^{(n+2)}(t)(x-t)^{n+1}dt$, which is exactly the claim at $n+1$.
Taking $n = N$ and recalling $a_k = f^{(k)}(0)/k!$ gives

$$f(x) - S_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\,dt,$$

which is the assertion of Theorem 8.3.1.

Problem 8.3.10. Let $f(x) = 1/\sqrt{1-x}$ on $(-1,1)$, with Taylor coefficients c_n as in Exercise 8.3.6 and partial sums S_N.
(a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1,1)$, and compute $E_2(x)$ for $x = 1/2$, $3/4$, and $8/9$.
(b) For a general x satisfying $-1 < x < 1$, show

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\,dt.$$

(c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \le |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking that this overestimate is in fact larger than $|E_2(x)|$ at the three computed values from part (a).
(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1, 1)$.

Solution. (a) $S_2(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2$, and

$$\begin{aligned} E_2(1/2) &= \sqrt{2} - \tfrac{43}{32} \approx 0.0705, \\ E_2(3/4) &= 2 - \tfrac{203}{128} = \tfrac{53}{128} \approx 0.4141, \\ E_2(8/9) &= 3 - \tfrac{47}{27} = \tfrac{34}{27} \approx 1.2593. \end{aligned}$$

The sketch: both curves pass through $(0, 1)$ and agree to second order there; on $(-1, 0]$ the parabola S_2 hugs f closely, while as $x \to 1^-$ the graph of f shoots off to $+\infty$ and S_2 climbs only to $S_2(1) = 15/8$, so the gap $E_2(x)$ widens without bound.
(b) By Theorem 8.3.1 with $N = 2$, using $f^{(3)}(t) = \frac{1\cdot3\cdot5}{2^3}(1-t)^{-7/2} = \frac{15}{8}(1-t)^{-7/2}$ (computed in 8.3.8),

$$\begin{aligned} E_2(x) &= \frac{1}{2!}\int_0^x \frac{15}{8}\,\frac{(x-t)^2}{(1-t)^{7/2}}\,dt \\ &= \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{dt}{(1-t)^{3/2}}. \end{aligned}$$

The hypotheses of Theorem 8.3.1 hold: f is infinitely differentiable on $(-R, R)$ for any $R < 1$, so $f^{(3)}$ is continuous there.
(c) For t between 0 and x (with $|x| < 1$), so that $1 - t > 0$:
(i) $0 \le t \le x < 1$. Then $x - t \ge 0$ and $x - t \le x(1-t)$ is equivalent to $xt \le t$, true since $t \ge 0$ and $x \le 1$.
(ii) $x < t \le 0$. Then $|x - t| = t - x$ and $t - x \le -x(1-t) = |x|(1-t)$ is equivalent to $t(1-x) \le 0$, true since $t \le 0$ and $x < 1$.
Hence $\left|\frac{x-t}{1-t}\right| \le |x|$ throughout the interval of integration. Pulling the bound out and using $\int_0^x (1-t)^{-3/2}dt = 2(1-x)^{-1/2} - 2$,

$$|E_2(x)| \;\le\; \frac{15}{16}x^2\left|\frac{2}{\sqrt{1-x}} - 2\right| = \frac{15}{8}x^2\left|\frac{1}{\sqrt{1-x}} - 1\right|.$$

Checking the three values (all positive x, so the absolute value bars drop):

$$\begin{aligned} x = \tfrac{1}{2}: &\quad \tfrac{15}{32}(\sqrt{2} - 1) \approx 0.1942 > 0.0705, \\ x = \tfrac{3}{4}: &\quad \tfrac{15}{8}\cdot\tfrac{9}{16} = \tfrac{135}{128} \approx 1.0547 > 0.4141, \\ x = \tfrac{8}{9}: &\quad \tfrac{15}{8}\cdot\tfrac{64}{81}\cdot 2 = \tfrac{80}{27} \approx 2.963 > 1.2593. \end{aligned}$$

(d) From $f^{(N+1)}(t) = (N+1)!\,c_{N+1}(1-t)^{-(2N+3)/2}$ and Theorem 8.3.1,

$$E_N(x) = (N+1)c_{N+1}\int_0^x \left(\frac{x-t}{1-t}\right)^N \frac{dt}{(1-t)^{3/2}},$$

so the estimate of part (c), together with $c_{N+1} \le c_0 = 1$, gives

$$|E_N(x)| \;\le\; K_x\,(N+1)|x|^N, \qquad K_x := 2\left|\frac{1}{\sqrt{1-x}} - 1\right|.$$

Since $|x| < 1$, $(N+1)|x|^N \to 0$ (the ratio of consecutive terms tends to $|x| < 1$), and K_x is a constant. Therefore $E_N(x) \to 0$, and equation (4) holds for every $x \in (-1, 1)$.

8.7 Exercises 8.3.11–8.4.4

Problem 8.3.11. Substituting x^2 for x in equation (4) gives

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} c_n x^{2n} \qquad \text{for all } |x| < 1.$$

Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

$$(5) \qquad \arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \qquad \text{for all } |x| < 1.$$

Solution. Term-by-term antidifferentiation (Exercise 6.5.4) applied to the power series for $1/\sqrt{1-x^2}$, followed by matching the two antiderivatives at $x = 0$.

In detail. The substitution is legitimate: for $|x| < 1$ we have $|x^2| < 1$, so equation (4) (established for all of $(-1, 1)$ in Exercise 8.3.10(d)) may be evaluated at x^2, and the resulting power series $\sum_{n=0}^{\infty} c_n x^{2n}$ (a genuine power series, with the odd coefficients equal to 0) converges on $(-1, 1)$.

Exercise 6.5.4 states that if $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$, then

$$G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also converges on $(-R, R)$ and satisfies $G'(x) = g(x)$ there. With $R = 1$ and $a_{2n} = c_n$, $a_{2n+1} = 0$, the antiderivative is

$$G(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}, \qquad G'(x) = \frac{1}{\sqrt{1-x^2}} \text{ on } (-1, 1).$$

By hypothesis arcsin has the same derivative on $(-1, 1)$, so $(G - \arcsin)' \equiv 0$ there; by Corollary 5.3.4 the difference is constant on the interval. Evaluating at $x = 0$ gives $G(0) = 0 = \arcsin(0)$, so the constant is 0 and $G(x) = \arcsin(x)$ for all $|x| < 1$, which is (5).

Problem 8.3.12. Our work thus far shows that the Taylor series in (5),

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1},$$

is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \le 1$. Carefully explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1, 1]$.

Solution. The Weierstrass M-Test with $M_n = c_n/(2n + 1)$, plus a continuity argument at the two endpoints.

First, $\sum_{n=0}^{\infty} M_n$ converges. Exercise 8.3.5, rephrased in Section 8.3, says

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{1}{c_n\sqrt{n}}, \qquad \text{i.e.} \qquad c_n\sqrt{n} \to \frac{1}{\sqrt{\pi}}.$$

A convergent sequence is bounded (Theorem 2.3.2), so there is $C > 0$ with $c_n \le C/\sqrt{n}$ for all $n \ge 1$, whence

$$M_n = \frac{c_n}{2n+1} \le \frac{C}{2n^{3/2}}.$$

Since $\sum n^{-3/2}$ converges (Corollary 2.4.7, $p = 3/2 > 1$), the Comparison Test (Theorem 2.7.4) gives convergence of $\sum M_n$.

Next, for $|x| \le 1$,

$$\left| \frac{c_n}{2n+1} x^{2n+1} \right| \le \frac{c_n}{2n+1} = M_n,$$

so by the Weierstrass M-Test (Corollary 6.4.5) the series converges uniformly on $[-1,1]$ to some function g.

Finally $g = \arcsin$ on all of $[-1,1]$. Each partial sum is a polynomial, hence continuous, so g is continuous on $[-1,1]$ by the Continuous Limit Theorem (Theorem 6.2.6). By Exercise 8.3.11, $g(x) = \arcsin(x)$ for $|x| < 1$. Since arcsin is continuous on $[-1,1]$ as well, letting $x \to 1^-$ along $(-1,1)$ and using sequential continuity of both functions gives

$$g(1) = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \arcsin(x) = \arcsin(1),$$

and symmetrically $g(-1) = \arcsin(-1)$. Thus the series in (5) converges uniformly to arcsin on $[-1,1]$.

Problem 8.3.13. Substituting $x = \sin(\theta)$ in (5), with $-\pi/2 \le \theta \le \pi/2$, gives

$$\theta = \arcsin(\sin(\theta)) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta),$$

which converges uniformly on $[-\pi/2, \pi/2]$.

(a) Show

$$\int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Wallis's product, where $b_n = \int_0^{\pi/2} \sin^n(x)\, dx$.

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Solution. (a) Uniform convergence licenses term-by-term integration, via the Integrable Limit Theorem (Theorem 7.4.4).

Write $h_N(x) = \sum_{n=0}^{N} \frac{c_n}{2n+1} x^{2n+1}$ and $g_N(\theta) = h_N(\sin\theta)$. Exercise 8.3.12 gives $h_N \to \arcsin$ uniformly on $[-1,1]$, and since sin maps $[-\pi/2, \pi/2]$ into $[-1,1]$ with $\arcsin(\sin\theta) = \theta$ there,

$$\sup_{|\theta| \le \pi/2} |g_N(\theta) - \theta| \;\le\; \sup_{|x| \le 1} |h_N(x) - \arcsin(x)| \longrightarrow 0,$$

so $g_N \to \theta$ uniformly on $[0, \pi/2]$ as well. Each g_N is continuous, hence integrable there (Theorem 7.2.9), so Theorem 7.4.4 applies:

$$\int_0^{\pi/2} \theta \, d\theta = \lim_{N \to \infty} \int_0^{\pi/2} g_N(\theta)\, d\theta.$$

Linearity of the integral over a finite sum (Theorem 7.4.2) gives

$$\begin{aligned} \int_0^{\pi/2} g_N &= \sum_{n=0}^{N} \frac{c_n}{2n+1} \int_0^{\pi/2} \sin^{2n+1}(\theta)\, d\theta \\ &= \sum_{n=0}^{N} \frac{c_n}{2n+1} b_{2n+1}, \end{aligned}$$

and letting $N \to \infty$ yields

$$\int_0^{\pi/2} \theta\, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1}.$$

(b) The two Wallis quantities collapse. From Exercise 8.3.3(c), with $b_1 = 1$,

$$b_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}, \qquad c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)},$$

so the even factors cancel and

$$c_n\, b_{2n+1} = \frac{1 \cdot 3 \cdots (2n-1)}{3 \cdot 5 \cdots (2n+1)} = \frac{1}{2n+1}$$

(the numerator and denominator share every factor except the 1 on top and the $2n+1$ below; for $n = 0$ both are empty products and the value is 1). Since $\int_0^{\pi/2} \theta\, d\theta = \pi^2/8$, part (a) becomes

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{c_n b_{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Now let $S = \sum_{n=1}^{\infty} 1/n^2$, which converges by Corollary 2.4.7. Because the series has nonnegative terms and converges, it converges absolutely, so splitting it into its odd- and even-indexed subseries is legitimate and

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{S}{4}. \end{aligned}$$

Hence $\frac{3}{4}S = \pi^2/8$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = S = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Problem 8.4.1. For $n \in \mathbf{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.
(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbf{N}$, and revisit part (a).

Solution. (a) $0\# = 0$, $(-2)\# = 1$, and $\frac{7}{2}\# = \frac{63}{8}$.
The defining sum obeys the recursion $n\# = n + (n-1)\#$, and this is the only structural feature of $\#$ available on $\mathbf{N}$, so run it backwards: $(n-1)\# = n\# - n$. Starting from $1\# = 1$,

$$\begin{aligned} 0\# &= 1\# - 1 = 0, \\ (-1)\# &= 0\# - 0 = 0, \\ (-2)\# &= (-1)\# - (-1) = 1. \end{aligned}$$

The recursion says nothing about $\frac{7}{2}\#$, since $\frac{7}{2}$ is not reachable from $\mathbf{N}$ by integer steps. Interpolating between $3\# = 6$ and $4\# = 10$ suggests a value near 8.
(b) Induction. The base case is $1\# = 1 = \frac{1}{2}(1)(2)$, and if $n\# = \frac{1}{2}n(n+1)$ then

$$(n+1)\# = (n+1) + n\# = (n+1) + \tfrac{1}{2}n(n+1) = \tfrac{1}{2}(n+1)(n+2).$$

Revisiting (a): the polynomial $p(x) = \frac{1}{2}x(x+1)$ satisfies $p(x) = x + p(x-1)$ for every real x, so it extends # to all of $\mathbf{R}$ while preserving the recursion. It returns $p(0) = 0$ and $p(-2) = \frac{1}{2}(-2)(-1) = 1$, matching the values forced above, and it supplies the missing value

$$\tfrac{7}{2}\# = \tfrac{1}{2} \cdot \tfrac{7}{2} \cdot \tfrac{9}{2} = \tfrac{63}{8} = 7.875,$$

in line with the guess.

Problem 8.4.2. Verify that the series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1}$$

converges absolutely for all $x \in \mathbf{R}$, that $E(x)$ is differentiable on $\mathbf{R}$, and $E'(x) = E(x)$.

Solution. The Ratio Test (Exercise 2.7.9) settles the first claim: for $x \neq 0$ the terms $a_n = x^n/n!$ are all nonzero, as that test requires, and

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} \\ &= \frac{|x|}{n+1} \longrightarrow 0 < 1,\end{aligned}$$

so $\sum |x^n/n!|$ converges; for $x = 0$ the series is $1 + 0 + 0 + \cdots$.
Hence the power series (1) converges at every point of $\mathbf{R}$, so it converges on $(-R, R)$ for every $R > 0$. By Theorem 6.5.7, E is differentiable on each such $(-R, R)$, and therefore at every point of $\mathbf{R}$ since each x lies in some $(-R, R)$, with derivative obtained by term-by-term differentiation:

$$\begin{aligned}E'(x) &= \sum_{n=1}^{\infty} n\,\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{m=0}^{\infty} \frac{x^m}{m!} = E(x),\end{aligned}$$

the reindexing $m = n - 1$ being a relabelling of the same terms.

Problem 8.4.3. (a) Use the results of Exercise 2.8.7 and the binomial formula to show that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.
(b) Show that $E(0) = 1$, $E(-x) = 1/E(x)$, and $E(x) > 0$ for all $x \in \mathbf{R}$.

Solution. (a) The Cauchy product of $E(x)$ with $E(y)$ is $E(x+y)$ term by term. Put $a_n = x^n/n!$ and $b_n = y^n/n!$; both series converge absolutely by Exercise 8.4.2, to $A = E(x)$ and $B = E(y)$. Exercise 2.8.7 therefore applies (after the harmless shift of its index set from $\{1, 2, \dots\}$ to $\{0, 1, \dots\}$) and gives

$$\sum_{k=0}^{\infty} d_k = AB = E(x)E(y), \qquad d_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

The binomial formula evaluates d_k:

$$\begin{aligned}d_k &= \sum_{j=0}^{k} \frac{x^j}{j!} \cdot \frac{y^{k-j}}{(k-j)!} = \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!}\, x^j y^{k-j} \\ &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} = \frac{(x+y)^k}{k!}.\end{aligned}$$

Hence $E(x)E(y) = \sum_{k=0}^{\infty} (x+y)^k/k! = E(x+y)$.

(b) $E(0) = 1$ since every term of (1) past $n = 0$ vanishes at $x = 0$ (with the convention $0^0 = 1$ for the constant term). Putting $y = -x$ in (a),

$$E(x)E(-x) = E(0) = 1,$$

so in particular $E(x) \neq 0$ for every x, and $E(-x) = 1/E(x)$.
For positivity: if $x \geq 0$, every term of (1) is nonnegative and the $n = 0$ term is 1, so $E(x) \geq 1 > 0$. If $x < 0$, then $-x > 0$ gives $E(-x) > 0$, and $E(x) = 1/E(-x) > 0$.

Problem 8.4.4. Define $e = E(1)$. Show $E(n) = e^n$ and $E(m/n) = (\sqrt[n]{e})^m$ for all $m, n \in \mathbf{Z}$.

Solution. Everything comes from $E(x + y) = E(x)E(y)$ (Exercise 8.4.3(a)) plus uniqueness of positive n-th roots. (The second identity of course requires $n \neq 0$; for $n < 0$ read $\sqrt[n]{e} = 1/\sqrt[-n]{e}$, which reduces that case to $m/n = (-m)/(-n)$ with $-n \in \mathbf{N}$. So take $n \in \mathbf{N}$ throughout.)
First identity. For $n \geq 0$, induct: $E(0) = 1 = e^0$ by Exercise 8.4.3(b), and

$$E(n + 1) = E(n)E(1) = e^n \cdot e = e^{n+1}.$$

For $n < 0$, Exercise 8.4.3(b) gives $E(n) = 1/E(-n) = 1/e^{-n} = e^n$.
Second identity. Fix $m \in \mathbf{Z}$ and $n \in \mathbf{N}$, and set $s = E(m/n)$, which is positive by Exercise 8.4.3(b). Applying 8.4.3(a) $n - 1$ times,

$$s^n = E\big(\tfrac{m}{n}\big)^n = E\big(n \cdot \tfrac{m}{n}\big) = E(m) = e^m,$$

the last equality by the first identity.
Now let $r = \sqrt[n]{e}$: the continuous function $f(t) = t^n$ has $f(0) = 0 < e$ and $f(1 + e) \geq 1 + e > e$, so the Intermediate Value Theorem (Theorem 4.5.1) supplies $r \in (0, 1 + e)$ with $r^n = e$, and f is strictly increasing on $[0, \infty)$, so r is the only positive such root. That same monotonicity makes $t \mapsto t^n$ one-to-one on $(0, \infty)$, so

$$(r^m)^n = (r^n)^m = e^m = s^n, \qquad r^m > 0,\ s > 0,$$

forces $s = r^m$. That is, $E(m/n) = (\sqrt[n]{e})^m$.

8.8 Exercises 8.4.5–8.4.11

Problem 8.4.5. Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \ldots$. (Here the limit is in the sense of Definition 8.4.1: given $f : [a, \infty) \to \mathbf{R}$, $\lim_{x\to\infty} f(x) = L$ means that for all $\epsilon > 0$ there exists $M > a$ such that $x \geq M$ implies $|f(x) - L| < \epsilon$.) To get started notice that when $x \geq 0$, all the terms in

$$e^x = E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{1}$$

are positive.

Solution. The single term $k = n + 1$ of (1) does it: for $x > 0$ all terms are positive, so discarding the others leaves

$$e^x > \frac{x^{n+1}}{(n+1)!},$$

and since $e^{-x} = 1/e^x$ (Exercise 8.4.3(b)),

$$0 < x^n e^{-x} = \frac{x^n}{e^x} < x^n \cdot \frac{(n+1)!}{x^{n+1}} = \frac{(n+1)!}{x} \qquad (x > 0).$$

Given $\epsilon > 0$, take $M = \max\{1, (n+1)!/\epsilon\} > 0$. For every $x \geq M$,

$$\left|x^n e^{-x} - 0\right| < \frac{(n+1)!}{x} \leq \frac{(n+1)!}{M} \leq \epsilon,$$

which is exactly Definition 8.4.1 with $a = 0$ and $L = 0$.

Problem 8.4.6. (a) Explain why we know e^x has an inverse function (let's call it $\log x$) defined on the strictly positive real numbers and satisfying
(i) $\log(e^y) = y$ for all $y \in \mathbf{R}$, and
(ii) $e^{\log x} = x$, for all $x > 0$.
(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)
(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log x + \log y \qquad \text{for all } x, y > 0.$$

(d) For $t > 0$ and $n \in \mathbf{N}$, t^n has the usual interpretation as $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \qquad \text{for all } n \in \mathbf{N}. \tag{2}$$

Solution. (a) Because $E : \mathbf{R} \to (0, \infty)$ is a strictly increasing bijection.
One-to-one: $E' = E > 0$ everywhere (Exercises 8.4.2, 8.4.3(b)), and E is differentiable hence continuous (Theorem 5.2.3), so for $x < y$ the Mean Value Theorem (Theorem 5.3.2) gives $E(y) - E(x) = E'(c)(y - x) > 0$; thus E is strictly increasing and in particular injective.
Onto $(0, \infty)$*:* the range lies in $(0, \infty)$ by Exercise 8.4.3(b). Conversely fix $y > 0$. All terms of (1) are nonnegative for $x \geq 0$, so $E(x) \geq 1 + x$ there, and with $a = -1/y$, $b = y$,

$$E(b) \geq 1 + y > y, \qquad E(a) = \frac{1}{E(1/y)} \leq \frac{1}{1 + 1/y} < y.$$

Since $a < 0 < b$ and E is continuous, the Intermediate Value Theorem (Theorem 4.5.1) on $[a, b]$ yields x with $E(x) = y$.
A bijection has an inverse; call it $\log : (0, \infty) \to \mathbf{R}$. Properties (i) and (ii) are precisely the two composition identities $\log \circ E = \mathrm{id}_{\mathbf{R}}$ and $E \circ \log = \mathrm{id}_{(0,\infty)}$ defining it.
(b) Exercise 5.2.12. On any $[-M, M]$ the function E is one-to-one and differentiable with $E'(x) = E(x) \neq 0$, so its inverse is differentiable on the range with

$$(\log)'(y) = \frac{1}{E'(x)} = \frac{1}{E(x)} = \frac{1}{y}, \qquad y = E(x).$$

Every $y > 0$ is $E(x)$ for some x, and then y lies in $E([-M, M])$ for any $M > |x|$; hence $(\log x)' = 1/x$ on all of $(0, \infty)$.
(c) With $y > 0$ fixed, $x \mapsto xy$ maps $(0, \infty)$ into $(0, \infty)$, so the Chain Rule and (b) give

$$\frac{d}{dx} \log(xy) = \frac{1}{xy} \cdot y = \frac{1}{x} = \frac{d}{dx} \log x.$$

Since $(0, \infty)$ is an interval, Corollary 5.3.4 yields a constant k with $\log(xy) = \log x + k$ for all $x > 0$. Evaluating at $x = 1$, where $\log 1 = 0$ because $E(0) = 1$, gives $k = \log y$. Hence $\log(xy) = \log x + \log y$.
(d) Induction on n using (c): $\log(t^1) = \log t$, and

$$\log(t^{n+1}) = \log(t^n \cdot t) = \log(t^n) + \log t = (n+1) \log t.$$

Now apply (ii) to the positive number t^n:

$$t^n = e^{\log(t^n)} = e^{n \log t}.$$

Problem 8.4.7. Recall Definition 8.4.2: given $t > 0$, the exponential function t^x is defined by $t^x = e^{x \log t}$ for all $x \in \mathbf{R}$.
(a) Show $t^{m/n} = (\sqrt[n]{t}\,)^m$ for all $m, n \in \mathbf{N}$.
(b) Show $\log(t^x) = x \log t$, for all $t > 0$ and $x \in \mathbf{R}$.
(c) Show t^x is differentiable on $\mathbf{R}$ and find the derivative.

Solution. (a) The witness is $s = e^{(1/n)\log t}$, which is exactly $\sqrt[n]{t}$. Indeed $s > 0$ by Exercise 8.4.3(b), and $\log s = \frac{1}{n} \log t$ by 8.4.6(i), so formula (2) of Exercise 8.4.6(d) applied to s gives

$$s^n = e^{n \log s} = e^{\log t} = t,$$

the last step by 8.4.6(ii). A positive n-th root of t is unique (as $u \mapsto u^n$ is strictly increasing on $(0, \infty)$), so $s = \sqrt[n]{t}$. Now, using (2) once more,

$$t^{m/n} = e^{\frac{m}{n} \log t} = e^{m \log s} = s^m = \left(\sqrt[n]{t}\right)^m.$$

(b) Immediate from 8.4.6(i):

$$\log(t^x) = \log\left(e^{x \log t}\right) = x \log t.$$

(c) $(t^x)' = t^x \log t$. The map $x \mapsto x \log t$ is differentiable with derivative $\log t$, and E is differentiable on $\mathbf{R}$ with $E' = E$ (Exercise 8.4.2), so the Chain Rule applies to the composition $t^x = E(x \log t)$:

$$\frac{d}{dx}\, t^x = E'(x \log t) \cdot \log t = e^{x \log t} \log t = t^x \log t.$$

Problem 8.4.8. Inspired by the fact that $0! = 1$ and $1! = 1$, let $h(x)$ satisfy
(i) $h(x) = 1$ for all $0 \le x \le 1$, and
(ii) $h(x) = x\, h(x-1)$ for all $x \in \mathbf{R}$.
(a) Find a formula for $h(x)$ on $[1, 2]$, $[2, 3]$, and $[n, n+1]$ for arbitrary $n \in \mathbf{N}$.
(b) Now do the same for $[-1, 0]$, $[-2, -1]$, and $[-n, -n+1]$.
(c) Sketch h over the domain $[-4, 4]$.

Solution. (a) On $[n, n+1]$,

$$h(x) \;=\; \prod_{k=0}^{n-1} (x - k) \;=\; x(x-1)(x-2)\cdots(x-n+1),$$

the empty product being 1 when $n = 0$. Indeed on $[1, 2]$ we have $x - 1 \in [0, 1]$, so (ii) and (i) give $h(x) = x \cdot 1 = x$; on $[2, 3]$ we have $x - 1 \in [1, 2]$, so $h(x) = x(x-1)$; and if the formula holds on $[n, n+1]$ then for $x \in [n+1, n+2]$ the point $x - 1$ lies in $[n, n+1]$, whence

$$h(x) = x\, h(x-1) = x \prod_{k=0}^{n-1} (x - 1 - k) = \prod_{k=0}^{n} (x - k).$$

In particular $h(n) = n(n-1)\cdots 1 = n!$.
(b) Rewriting (ii) as $h(x) = h(x+1)/(x+1)$ and iterating downward, on $(-n, -n+1]$,

$$h(x) \;=\; \frac{1}{\prod_{k=1}^{n} (x+k)} \;=\; \frac{1}{(x+1)(x+2)\cdots(x+n)}.$$

For $x \in (-1, 0]$ the point $x + 1$ lies in $(0, 1]$, so $h(x+1) = 1$ and $h(x) = 1/(x+1)$; for $x \in (-2, -1]$ the point $x+1$ lies in $(-1, 0]$, so $h(x) = 1/\big((x+1)(x+2)\big)$; the induction continues exactly as in (a). Note h is not defined at $x = -1, -2, -3, \ldots$: setting $x = 0$ in (ii) forces $h(0) = 0 \cdot h(-1)$, which is incompatible with $h(0) = 1$, and each formula above has the factor $x + n$ vanishing as $x \to -n^+$.

(c) On $[0, 4]$ the graph is the flat segment $h \equiv 1$ on $[0, 1]$, then the line $h = x$ rising from 1 to 2 on $[1, 2]$, then the parabola $x(x-1)$ rising from 2 to 6 on $[2, 3]$, then the cubic $x(x-1)(x-2)$ rising from 6 to 24 on $[3, 4]$ — continuous, but with a visible corner at each positive integer (the left and right derivatives at $x = 2$, for instance, are 1 and 3).
On the negative side h has a vertical asymptote at every negative integer and alternates sign from one gap to the next, since on $(-n, -n+1)$ exactly $n-1$ of the factors $x+1, \ldots, x+n$ are negative:

interval	sign of h	sample value	behavior at the ends
$(-1, 0)$	$+$	$h(-1/2) = 2$	$\to +\infty$ at -1; $h(0) = 1$
$(-2, -1)$	$-$	$h(-3/2) = -4$	$\to -\infty$ at both ends
$(-3, -2)$	$+$	$h(-5/2) = 8/3$	$\to +\infty$ at both ends
$(-4, -3)$	$-$	$h(-7/2) = -16/15$	$\to -\infty$ at both ends

On $(-1, 0)$ the graph falls from $+\infty$ to the value 1 at the origin, where it meets the flat piece; each of the other three branches is a single arc pinned to $\pm\infty$ at both of its asymptotes, with one turning point near the middle of the gap (at $x = -3/2$ exactly on $(-2, -1)$).

Problem 8.4.9. (a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\epsilon > 0$ there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \epsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)
(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.
(c) Part (a) is a Cauchy criterion, and part (b) is a comparison test. State and prove an absolute convergence test for improper integrals.

Solution. (a) The criterion is exactly the Cauchy criterion for $F(b) = \int_a^b f$, which by Definition 8.4.3 is the quantity whose limit is at issue; note $\int_c^d f = F(d) - F(c)$ by additivity (Theorem 7.4.1).
($\Rightarrow$) Let $L = \lim_{b\to\infty} F(b)$ and pick $M > a$ with $|F(b) - L| < \epsilon/2$ for all $b \geq M$. Then for $d > c \geq M$,

$$\left| \int_c^d f \right| = |F(d) - F(c)| \leq |F(d) - L| + |L - F(c)| < \epsilon.$$

($\Leftarrow$) Assume the criterion and set $a_n = F(a+n)$. Given $\epsilon > 0$ take the corresponding M, and let $N \in \mathbf{N}$ satisfy $a + N \geq M$. For $m > n \geq N$,

$$|a_m - a_n| = \left| \int_{a+n}^{a+m} f \right| < \epsilon,$$

so (a_n) is Cauchy and converges to some L (Theorem 2.6.4). To upgrade this to $F(b) \to L$, let $\epsilon > 0$, take M for the value $\epsilon/2$, and choose N with both $a + N \geq M$ and $|a_N - L| < \epsilon/2$. For any $b \geq a + N$,

$$|F(b) - L| \leq \left| \int_{a+N}^b f \right| + |a_N - L| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

(the first term is 0 when $b = a + N$). Hence $\int_a^\infty f$ converges.
(b) Apply (a) to g: given $\epsilon > 0$ there is $M > a$ with $\left|\int_c^d g\right| < \epsilon$ whenever $d > c \geq M$. Since $0 \leq f \leq g$ on $[c, d]$, Theorem 7.4.2 (iv) gives

$$\left| \int_c^d f \right| = \int_c^d f \leq \int_c^d g = \left| \int_c^d g \right| < \epsilon,$$

the two absolute-value signs being removable because $f, g \geq 0$. Now (a), applied to f, yields convergence.

(c) Absolute convergence test: if $\int_a^\infty |f|$ converges, then $\int_a^\infty f$ converges. (Theorem 7.4.2 (v) supplies both the integrability of $|f|$ on each $[a,b]$ and the inequality used below.)
Given $\epsilon > 0$, part (a) applied to $|f|$ produces $M > a$ with $\int_c^d |f| < \epsilon$ for all $d > c \geq M$. Then Theorem 7.4.2 (v) gives $\left|\int_c^d f\right| \leq \int_c^d |f| < \epsilon$, and part (a) applied to f finishes the proof.

Problem 8.4.10. (a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t}\,dt = 1.$$

(b) Show

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\,dt, \qquad \text{for all } \alpha > 0. \tag{3}$$

Solution. (a) $\int_0^b e^{-t}\,dt = 1 - e^{-b} \to 1$.
In detail: e^{-t} is continuous, hence integrable on every $[0,b]$ (Theorem 7.2.9), so the improper integral is defined by Definition 8.4.3. Since $E' = E$ (Exercise 8.4.2), the Chain Rule (Theorem 5.2.5) gives $\frac{d}{dt}\big(-e^{-t}\big) = e^{-t}$, and the Fundamental Theorem of Calculus (Theorem 7.5.1 (i)) yields

$$\int_0^b e^{-t}\,dt = \big(-e^{-b}\big) - \big(-e^{0}\big) = 1 - e^{-b},$$

using $E(0) = 1$ (Exercise 8.4.3 (b)). By Exercise 8.4.5 with $n = 0$ we have $e^{-b} \to 0$, so the limit as $b \to \infty$ is 1.
(b) Identically, $\frac{d}{dt}\big(-\frac{1}{\alpha}e^{-\alpha t}\big) = e^{-\alpha t}$, so

$$\int_0^b e^{-\alpha t}\,dt = \frac{1 - e^{-\alpha b}}{\alpha}.$$

As $b \to \infty$ we have $\alpha b \to \infty$ (here $\alpha > 0$ is essential), so $e^{-\alpha b} \to 0$ by Exercise 8.4.5 again, and the limit is $1/\alpha$.

Problem 8.4.11. (a) Evaluate $\int_0^b te^{-\alpha t}\,dt$ using the integration-by-parts formula from Exercise 7.5.6. The result will be an expression in α and b.
(b) Now compute $\int_0^\infty te^{-\alpha t}\,dt$ and verify equation (4), namely

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\,dt.$$

Solution. (a)

$$\int_0^b te^{-\alpha t}\,dt \;=\; \frac{1}{\alpha^2}\Big(1 - e^{-\alpha b} - \alpha b\, e^{-\alpha b}\Big).$$

Apply Exercise 7.5.6 with $h(t) = t$ and $k(t) = -\frac{1}{\alpha}e^{-\alpha t}$, both of which have continuous derivatives on $[0,b]$ (the hypothesis of that exercise), so that $k'(t) = e^{-\alpha t}$ and $h'(t) = 1$:

$$\begin{aligned}\int_0^b te^{-\alpha t}\,dt &= h(b)k(b) - h(0)k(0) - \int_0^b h'(t)k(t)\,dt \\ &= -\frac{b}{\alpha}e^{-\alpha b} + \frac{1}{\alpha}\int_0^b e^{-\alpha t}\,dt \\ &= -\frac{b}{\alpha}e^{-\alpha b} + \frac{1}{\alpha}\cdot\frac{1 - e^{-\alpha b}}{\alpha},\end{aligned}$$

the last integral coming from Exercise 8.4.10 (b). Collecting over α^2 gives the displayed formula.

(b) Both error terms vanish: $e^{-\alpha b} \to 0$ and, writing $x = \alpha b$,

$$\alpha b\, e^{-\alpha b} = x e^{-x} \longrightarrow 0$$

by Exercise 8.4.5 with $n = 1$. Hence

$$\begin{aligned}\int_0^\infty t e^{-\alpha t}\, dt &= \lim_{b\to\infty} \frac{1}{\alpha^2}\Big(1 - e^{-\alpha b} - \alpha b\, e^{-\alpha b}\Big) \\ &= \frac{1}{\alpha^2},\end{aligned}$$

which is equation (4).

8.9 Exercises 8.4.12–8.4.18

Problem 8.4.12. Assume the function $f(x, t)$ is continuous on the rectangle

$$D = \{(x, t) : a \le x \le b,\ c \le t \le d\}.$$

Explain why the function

$$F(x) = \int_c^d f(x, t)\, dt$$

is properly defined for all $x \in [a, b]$.

Solution. Because each horizontal slice $t \mapsto f(x, t)$ is a continuous function on the compact interval $[c, d]$, hence integrable by Theorem 7.2.9.
The slice is continuous: fix $x \in [a, b]$ and $t_0 \in [c, d]$, let $\epsilon > 0$, and take the $\delta > 0$ supplied by Definition 8.4.4 at the point (x, t_0). If $|t - t_0| < \delta$ then

$$\|(x, t) - (x, t_0)\| = \sqrt{(x - x)^2 + (t - t_0)^2} = |t - t_0| < \delta,$$

so $|f(x, t) - f(x, t_0)| < \epsilon$. Thus the Riemann integral defining $F(x)$ exists for every $x \in [a, b]$, and $F : [a, b] \to \mathbf{R}$ is a well-defined function.

Problem 8.4.13. Prove Theorem 8.4.5: If $f(x, t)$ is continuous on $D = \{(x, t) : a \le x \le b,\ c \le t \le d\}$, then $F(x) = \int_c^d f(x, t)\, dt$ is uniformly continuous on $[a, b]$.

Solution. The point is that the δ from uniform continuity of f on D serves as the δ for F, unchanged. Assume $c < d$ (otherwise $F \equiv 0$ and there is nothing to prove), and note F is defined by Exercise 8.4.12. The rectangle D is closed and bounded, hence compact in $\mathbf{R}^2$, so the continuous function f is uniformly continuous on D — this is the $\mathbf{R}^2$ analogue of Theorem 4.4.7 recorded in the text preceding Theorem 8.4.5. Given $\epsilon > 0$, choose $\delta > 0$ so that

$$\|(x, t) - (y, s)\| < \delta \implies |f(x, t) - f(y, s)| < \frac{\epsilon}{2(d - c)}.$$

Now let $x, y \in [a, b]$ with $|x - y| < \delta$. For every $t \in [c, d]$ we have $\|(x, t) - (y, t)\| = |x - y| < \delta$, so

$$-\frac{\epsilon}{2(d - c)} \ \le\ f(x, t) - f(y, t) \ \le\ \frac{\epsilon}{2(d - c)}.$$

The function $t \mapsto f(x, t) - f(y, t)$ is integrable on $[c, d]$ and $F(x) - F(y)$ is its integral (Theorem 7.4.2 (i), (ii)), so Theorem 7.4.2 (iii) gives

$$\begin{aligned}|F(x) - F(y)| &= \left|\int_c^d \big(f(x, t) - f(y, t)\big)\, dt\right| \\ &\le \frac{\epsilon}{2(d - c)}\,(d - c) = \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Since δ was produced from ϵ alone, with no reference to x or y, F is uniformly continuous on $[a, b]$.

Problem 8.4.14. Finish the proof of Theorem 8.4.6: If $f(x,t)$ and $f_x(x,t)$ are continuous on $D = \{(x,t) : a \le x \le b,\ c \le t \le d\}$, then $F(x) = \int_c^d f(x,t)\,dt$ is differentiable and

$$F'(x) = \int_c^d f_x(x,t)\,dt.$$

(The proof has begun: fix x in $[a,b]$ and let $\epsilon > 0$ be arbitrary; the task is to produce $\delta > 0$ such that

$$\left| \frac{F(z) - F(x)}{z - x} - \int_c^d f_x(x,t)\,dt \right| < \epsilon \tag{5}$$

whenever $0 < |z - x| < \delta$.)

Solution. The Mean Value Theorem converts the difference quotient into a value of f_x, and uniform continuity of f_x on the compact rectangle D then makes that value uniformly close to $f_x(x,t)$.
Assume $c < d$ (else both sides of (5) are 0). Since f_x is continuous on the compact set D, it is uniformly continuous there (the $\mathbf{R}^2$ analogue of Theorem 4.4.7 quoted before Theorem 8.4.5); choose $\delta > 0$ so that

$$\|(u,s) - (v,t)\| < \delta \implies |f_x(u,s) - f_x(v,t)| < \frac{\epsilon}{2(d-c)}.$$

Let $z \in [a,b]$ satisfy $0 < |z - x| < \delta$. Both $F(z)$ and $F(x)$ exist by Exercise 8.4.12, and by linearity (Theorem 7.4.2 (i), (ii)),

$$\frac{F(z) - F(x)}{z - x} - \int_c^d f_x(x,t)\,dt = \int_c^d \left[\frac{f(z,t) - f(x,t)}{z - x} - f_x(x,t) \right] dt,$$

the integrand being continuous in t — each of $t \mapsto f(z,t)$, $t \mapsto f(x,t)$ and $t \mapsto f_x(x,t)$ is, by the slice argument of Exercise 8.4.12 — hence integrable by Theorem 7.2.9.
Fix $t \in [c,d]$. By hypothesis $s \mapsto f(s,t)$ is differentiable on $[a,b]$, hence continuous there, so the Mean Value Theorem (Theorem 5.3.2), applied on the closed interval with endpoints x and z, produces a point $\xi = \xi(z,t)$ strictly between x and z with

$$\frac{f(z,t) - f(x,t)}{z - x} = f_x(\xi, t).$$

Because ξ lies between x and z we have $\|(\xi,t) - (x,t)\| = |\xi - x| < |z - x| < \delta$, so the choice of δ gives

$$\left| \frac{f(z,t) - f(x,t)}{z - x} - f_x(x,t) \right| = |f_x(\xi,t) - f_x(x,t)| < \frac{\epsilon}{2(d-c)}$$

for every $t \in [c,d]$. Integrating this bound over $[c,d]$ (Theorem 7.4.2 (iii)) yields

$$\left| \frac{F(z) - F(x)}{z - x} - \int_c^d f_x(x,t)\,dt \right| \le \frac{\epsilon}{2(d-c)}\,(d-c) = \frac{\epsilon}{2} < \epsilon,$$

which is (5). As $\epsilon > 0$ was arbitrary,

$$\lim_{z \to x} \frac{F(z) - F(x)}{z - x} = \int_c^d f_x(x,t)\,dt,$$

so F is differentiable at x with $F'(x) = \int_c^d f_x(x,t)\,dt$. Since $x \in [a,b]$ was arbitrary, the theorem is proved.

Problem 8.4.15. (a) Show that the improper integral $\int_0^\infty e^{-xt}\,dt$ converges uniformly to $1/x$ on the set $[1/2,\infty)$.
(b) Is the convergence uniform on $(0,\infty)$?

Solution. (a) The tail is e^{-xd}/x, and $\sup_{x\geq 1/2} e^{-xd}/x = 2e^{-d/2} \to 0$.
Indeed for $x>0$ and $d>0$,

$$\int_0^d e^{-xt}\,dt = \left[\frac{-e^{-xt}}{x}\right]_0^d = \frac{1-e^{-xd}}{x},$$

so that (letting $d\to\infty$) the improper integral converges to $1/x$ for each $x>0$, with

$$\left|\frac{1}{x} - \int_0^d e^{-xt}\,dt\right| = \frac{e^{-xd}}{x}.$$

For $x \geq 1/2$ we have $1/x \leq 2$ and $e^{-xd} \leq e^{-d/2}$, so given $\epsilon > 0$ the choice $M = \max\{1,\, 2\log(4/\epsilon)\} > c = 0$ gives $2e^{-M/2} \leq \epsilon/2$ and hence

$$\left|\frac{1}{x} - \int_0^d e^{-xt}\,dt\right| \leq 2e^{-d/2} \leq 2e^{-M/2} < \epsilon$$

for all $d \geq M$ and all $x \geq 1/2$, which is Definition 8.4.7.
(b) No. Take $\epsilon = 1$; given any $M>0$, set $d = \max\{M,1\}$ and $x = 1/(2d) \in (0,\infty)$. Then $d \geq M$ and

$$\left|\frac{1}{x} - \int_0^d e^{-xt}\,dt\right| = \frac{e^{-xd}}{x} = 2d\,e^{-1/2} \geq 2e^{-1/2} > 1,$$

so no single M serves all of $(0,\infty)$.

Problem 8.4.16. Prove the following analogue of the Weierstrass M-Test for improper integrals: If $f(x,t)$ satisfies $|f(x,t)| \leq g(t)$ for all $x \in A$ and $\int_a^\infty g(t)\,dt$ converges, then $\int_a^\infty f(x,t)\,dt$ converges uniformly on A.

Solution. The Cauchy criterion of Exercise 8.4.9(a) applied to g produces a single M that works for every $x \in A$ at once.
First, Definition 8.4.7 requires that $F(x) = \int_a^\infty f(x,t)\,dt$ exist for each $x \in A$: it does, since $0 \leq |f(x,t)| \leq g(t)$ and $\int_a^\infty g$ converges, so the absolute convergence test of Exercise 8.4.9(c) applies.
Now let $\epsilon > 0$. The Cauchy criterion of Exercise 8.4.9(a), applied to the convergent $\int_a^\infty g$, supplies $M > a$ with

$$\left|\int_u^v g\right| = \int_u^v g < \frac{\epsilon}{2} \qquad \text{whenever } v > u \geq M.$$

Fix $d \geq M$ and $x \in A$. For every $b > d$,

$$\begin{aligned}\left|\int_d^b f(x,t)\,dt\right| &\leq \int_d^b |f(x,t)|\,dt \\ &\leq \int_d^b g(t)\,dt < \frac{\epsilon}{2},\end{aligned}$$

and letting $b\to\infty$ in the outer inequality gives

$$\begin{aligned}\left|F(x) - \int_a^d f(x,t)\,dt\right| &= \left|\lim_{b\to\infty}\int_d^b f(x,t)\,dt\right| \\ &\leq \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

The bound holds for all $d \geq M$ and all $x \in A$, with M chosen independently of x, which is exactly Definition 8.4.7.

Problem 8.4.17. Prove Theorem 8.4.8: If $f(x,t)$ is continuous on $D = \{(x,t) : a \le x \le b,\ c \le t\}$, then

$$F(x) = \int_c^\infty f(x,t)\,dt$$

is uniformly continuous on $[a,b]$, provided the integral converges uniformly.

Solution. Set $F_n(x) = \int_c^{c+n} f(x,t)\,dt$; then $F_n \to F$ uniformly on $[a,b]$ with each F_n continuous, so Theorem 6.2.6 and Theorem 4.4.7 finish the job.
Continuity of F_n: the restriction of f to the rectangle $\{(x,t) : a \le x \le b,\ c \le t \le c+n\}$ is continuous, which is exactly the hypothesis of Theorem 8.4.5 there.
Uniformity of $F_n \to F$: let $\epsilon > 0$, and let Definition 8.4.7 supply $M > c$ with

$$\left| F(x) - \int_c^d f(x,t)\,dt \right| < \epsilon \qquad \text{for all } d \ge M,\ x \in [a,b].$$

Choose $N \in \mathbf{N}$ with $c + N \ge M$. For $n \ge N$ the value $d = c+n$ satisfies $d \ge M$, so

$$|F(x) - F_n(x)| < \epsilon \qquad \text{for all } x \in [a,b],$$

with N independent of x.
Theorem 6.2.6 now makes the uniform limit F continuous on $[a,b]$, and $[a,b]$ is compact, so Theorem 4.4.7 upgrades this to uniform continuity.

Problem 8.4.18. Prove Theorem 8.4.9: Assume the function $f(x,t)$ is continuous on $D = \{(x,t) : a \le x \le b,\ c \le t\}$ and that $F(x) = \int_c^\infty f(x,t)\,dt$ exists for each $x \in [a,b]$. If the derivative function $f_x(x,t)$ exists and is continuous, then

$$F'(x) = \int_c^\infty f_x(x,t)\,dt,$$

this last being equation (7) of the section, provided the integral in (7) converges uniformly.

Solution. Truncate and apply the Differentiable Limit Theorem (Theorem 6.3.1) to $F_n(x) = \int_c^{c+n} f(x,t)\,dt$.
Write also

$$G(x) = \int_c^\infty f_x(x,t)\,dt, \qquad G_n(x) = \int_c^{c+n} f_x(x,t)\,dt.$$

On the rectangle $\{(x,t) : a \le x \le b,\ c \le t \le c+n\}$ both f and f_x are continuous, so Theorem 8.4.6 applies there and yields that F_n is differentiable on $[a,b]$ with

$$F_n'(x) = \int_c^{c+n} f_x(x,t)\,dt = G_n(x).$$

The hypothesis that $F(x)$ exists for each $x \in [a,b]$ says exactly that $F_n \to F$ pointwise on $[a,b]$ (Definition 8.4.3, along the subsequence of endpoints $d = c+n$, which suffices since the full limit as $d \to \infty$ exists).
The convergence $G_n \to G$ is uniform on $[a,b]$: given $\epsilon > 0$, uniform convergence of the integral in (7) gives $M > c$ with $|G(x) - \int_c^d f_x(x,t)\,dt| < \epsilon$ for all $d \ge M$ and all $x \in [a,b]$; taking N with $c + N \ge M$ gives $|G(x) - G_n(x)| < \epsilon$ for all $n \ge N$ and all $x \in [a,b]$.
So Theorem 6.3.1 applies on the closed interval $[a,b]$: the F_n are differentiable, $F_n \to F$ pointwise, and $F_n' = G_n \to G$ uniformly. Hence F is differentiable and

$$F'(x) = G(x) = \int_c^\infty f_x(x,t)\,dt.$$

8.10 Exercises 8.4.19–8.5.2

Problem 8.4.19. Recall equation (3) of this section: $\dfrac{1}{\alpha} = \displaystyle\int_0^\infty e^{-\alpha t}\,dt$ for all $\alpha > 0$.
(a) Although we verified it directly, show how to use the theorems in this section to give a second justification for the formula

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\,dt, \qquad \text{for all } \alpha > 0.$$

(b) Now derive the formula

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\,dt, \qquad \text{for all } \alpha > 0,$$

which is equation (8) of the section.

Solution. Differentiate (3) under the integral sign, justified by Theorem 8.4.9 with the uniform convergence supplied by the M-Test of Exercise 8.4.16.
Lemma (used twice below): for an integer $m \geq 1$ and $\beta > 0$, the sup $C_m = \sup_{t\geq 0} t^m e^{-\beta t/2}$ is finite, being attained at $t = 2m/\beta$ with value $(2m/e\beta)^m$; so $t^m e^{-\beta t} \leq C_m e^{-\beta t/2}$ and $\int_0^\infty t^m e^{-\beta t}\,dt$ converges by (3) and the comparison test of Exercise 8.4.9(b).
(a) Fix $0 < a < b$ and apply Theorem 8.4.9 with α as the parameter, $c = 0$, and $f(\alpha, t) = e^{-\alpha t}$ on $D = \{(\alpha, t) : a \leq \alpha \leq b,\ 0 \leq t\}$. Its hypotheses hold: f is continuous on D; $F(\alpha) = 1/\alpha$ exists for each $\alpha \in [a, b]$ by (3); $f_\alpha(\alpha, t) = -te^{-\alpha t}$ exists and is continuous on D; and

$$|-te^{-\alpha t}| \leq te^{-at} \qquad (a \leq \alpha \leq b,\ t \geq 0),$$

whose integral converges by the lemma ($m = 1$, $\beta = a$), so Exercise 8.4.16 makes $\int_0^\infty f_\alpha$ uniformly convergent on $[a, b]$. Hence

$$-\frac{1}{\alpha^2} = \left(\frac{1}{\alpha}\right)' = \int_0^\infty -te^{-\alpha t}\,dt,$$

and $0 < a < b$ were arbitrary, so $1/\alpha^2 = \int_0^\infty te^{-\alpha t}\,dt$ for every $\alpha > 0$.
(b) Induction on n, the case $n = 0$ being (3). Assume (8) for n and fix $0 < a < b$. Apply Theorem 8.4.9 to $f(\alpha, t) = t^n e^{-\alpha t}$ on $\{a \leq \alpha \leq b,\ 0 \leq t\}$: it is continuous, $F(\alpha) = n!/\alpha^{n+1}$ exists by the inductive hypothesis, and

$$f_\alpha(\alpha, t) = -t^{n+1}e^{-\alpha t}$$

is continuous with $|f_\alpha(\alpha, t)| \leq t^{n+1}e^{-at}$, whose improper integral converges by the lemma ($m = n + 1$, $\beta = a$). Exercise 8.4.16 gives uniform convergence on $[a, b]$, so Theorem 8.4.9 applies:

$$\begin{aligned}\int_0^\infty -t^{n+1}e^{-\alpha t}\,dt &= \frac{d}{d\alpha}\left(\frac{n!}{\alpha^{n+1}}\right) \\ &= -\frac{(n+1)\,n!}{\alpha^{n+2}} = -\frac{(n+1)!}{\alpha^{n+2}}.\end{aligned}$$

Negating gives (8) for $n + 1$, and again a, b were arbitrary, so the identity holds for all $\alpha > 0$.

Problem 8.4.20. Recall Definition 8.4.10: for $x \geq 0$ the factorial function is $x! = \displaystyle\int_0^\infty t^x e^{-t}\,dt$, where $t^x = e^{x \log t}$ for $t > 0$.
(a) Show that $x!$ is an infinitely differentiable function on $(0, \infty)$ and produce a formula for the nth derivative. In particular show that $(x!)'' > 0$.
(b) Use the integration-by-parts formula employed earlier to show that $x!$ satisfies the functional

equation

$$(x+1)! = (x+1)\, x!.$$

Solution. (a) The formula is

$$(x!)^{(n)} = \int_0^\infty (\log t)^n\, t^x e^{-t}\, dt, \qquad x > 0,\ n \geq 0.$$

Fix $0 < a < b$ and work on $D = \{(x,t) : a \leq x \leq b,\ 0 \leq t\}$. Put

$$h_n(x,t) = (\log t)^n\, t^x e^{-t} \quad (t > 0), \qquad h_n(x,0) = 0.$$

Each h_n is continuous on D (for $t \to 0^+$, $|\log t|^n t^x \leq |\log t|^n t^a \to 0$, so 0 is the continuous value at $t = 0$), and $\partial h_n/\partial x = h_{n+1}$ is continuous on D for the same reason.
Domination for the M-Test: with $M_n = \sup_{0<t\leq 1} |\log t|^n t^a < \infty$, and $K_n = \sup_{t\geq 1} t^{n+b} e^{-t/2} < \infty$ handling $t \geq 1$ via $\log t \leq t$ and $t^x \leq t^b$ there, the function

$$g_n(t) = \begin{cases} M_n, & 0 \leq t \leq 1, \\ K_n\, e^{-t/2}, & t > 1, \end{cases}$$

satisfies $|h_n(x,t)| \leq g_n(t)$ for all $x \in [a,b]$, $t \geq 0$, and

$$\int_0^\infty g_n = M_n + 2K_n e^{-1/2} < \infty.$$

By Exercise 8.4.16, $\int_0^\infty h_n(x,t)\, dt$ converges uniformly on $[a,b]$ for every $n \geq 0$.
Now induct, the case $n = 0$ being Definition 8.4.10. Given $(x!)^{(n)} = \int_0^\infty h_n(x,t)\, dt$ on $[a,b]$, Theorem 8.4.9 applies to $f = h_n$ (continuous on D; integral existing at each $x \in [a,b]$; x-derivative h_{n+1} continuous; $\int_0^\infty h_{n+1}$ uniformly convergent by the last paragraph), giving

$$(x!)^{(n+1)} = \frac{d}{dx} \int_0^\infty h_n(x,t)\, dt = \int_0^\infty (\log t)^{n+1} t^x e^{-t}\, dt.$$

Since $0 < a < b$ were arbitrary, $x!$ has derivatives of every order on $(0,\infty)$.
The integrand $(\log t)^2 t^x e^{-t}$ is nonnegative on $[0,\infty)$ and continuous and strictly positive on $[2,3]$, so

$$(x!)'' = \int_0^\infty (\log t)^2 t^x e^{-t}\, dt \ \geq \int_2^3 (\log t)^2 t^x e^{-t}\, dt > 0.$$

(b) Integrate by parts on $[0,d]$ with $u = t^{x+1}$, $dv = e^{-t} dt$:

$$\begin{aligned} \int_0^d t^{x+1} e^{-t}\, dt &= \left[-t^{x+1} e^{-t}\right]_0^d + (x+1) \int_0^d t^x e^{-t}\, dt \\ &= -d^{x+1} e^{-d} + (x+1) \int_0^d t^x e^{-t}\, dt. \end{aligned}$$

For $d \geq 1$ and any integer $m \geq x+1$, the lemma of Exercise 8.4.19 with $\beta = 1$ gives $d^{x+1} e^{-d} \leq d^m e^{-d} \leq C_m e^{-d/2} \to 0$, so letting $d \to \infty$ gives

$$(x+1)! = \int_0^\infty t^{x+1} e^{-t}\, dt = (x+1) \int_0^\infty t^x e^{-t}\, dt = (x+1)\, x!.$$

Problem 8.4.21. (Completion of the proof of the Bohr-Mollerup Theorem, Theorem 8.4.11.) Let f be a positive function on $x \geq 0$ with (i) $f(0) = 1$, (ii) $f(x+1) = (x+1) f(x)$, and (iii) $\log(f(x))$ convex; so $f(n) = n!$ for all $n \in \mathbf{N}$. The proof uses the fact that if $[a,b]$ and $[a',b']$ lie in the domain of a convex function ϕ with $a \le a'$ and $b \le b'$, then

$$\frac{\phi(b)-\phi(a)}{b-a} \le \frac{\phi(b')-\phi(a')}{b'-a'}.$$

Fix $n \in \mathbf{N}$ and $x \in (0,1]$.
(a) Use the convexity of $\log(f(x))$ and the three intervals $[n-1,n]$, $[n,n+x]$, and $[n,n+1]$ to show

$$x\log(n) \le \log(f(n+x)) - \log(n!) \le x\log(n+1).$$

(b) Show $\log(f(n+x)) = \log(f(x)) + \log((x+1)(x+2)\cdots(x+n))$.
(c) Now establish that

$$0 \le \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \le x\log\left(1+\frac{1}{n}\right).$$

(d) Conclude that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}, \qquad \text{for all } x \in (0,1].$$

(e) Finally, show that the conclusion in (d) holds for all $x \ge 0$.

Solution. Write $\phi = \log f$, which is convex by (iii), and recall $\phi(n) = \log(n!)$.
(a) Two applications of the chord-slope inequality. First take $[a,b] = [n-1,n]$ and $[a',b'] = [n,n+x]$; here $n-1 \le n$ and $n \le n+x$, so

$$\log n = \frac{\phi(n)-\phi(n-1)}{n-(n-1)} \le \frac{\phi(n+x)-\phi(n)}{x}.$$

(The left side is $\log(n!) - \log((n-1)!) = \log n$.) Second take $[a,b] = [n,n+x]$ and $[a',b'] = [n,n+1]$; here $n \le n$ and $n+x \le n+1$ because $x \le 1$, so

$$\frac{\phi(n+x)-\phi(n)}{x} \le \frac{\phi(n+1)-\phi(n)}{1} = \log(n+1),$$

the right side being $\log((n+1)!) - \log(n!)$. Multiplying the chain by $x > 0$:

$$x\log(n) \le \log(f(n+x)) - \log(n!) \le x\log(n+1).$$

(b) Iterating (ii) n times from $f(x)$ upward,

$$f(n+x) = (x+n)(x+n-1)\cdots(x+1)\,f(x),$$

and all factors are positive, so taking logarithms gives

$$\log(f(n+x)) = \log(f(x)) + \log\big((x+1)(x+2)\cdots(x+n)\big).$$

(c) Substitute (b) into (a) and subtract $x\log n$ throughout:

$$\begin{aligned} 0 &\le \log(f(x)) + \log\Big(\prod_{k=1}^{n}(x+k)\Big) - \log(n!) - x\log n \\ &\le x\log(n+1) - x\log(n). \end{aligned}$$

The middle expression is exactly

$$\log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right),$$

and the right-hand bound is $x\log((n+1)/n) = x\log(1+1/n)$, as claimed.
(d) Since $x\log(1+1/n) \to 0$ as $n\to\infty$, squeezing the quantity in (c) between 0 and $x\log(1+1/n)$ gives

$$\lim_{n\to\infty}\left[\log(f(x)) - \log\left(\frac{n^x n!}{(x+1)\cdots(x+n)}\right)\right] = 0.$$

Thus $\log G_n(x) \to \log(f(x))$ where $G_n(x) = \dfrac{n^x n!}{(x+1)\cdots(x+n)}$, and applying the continuous function exp (Theorem 4.3.2, continuity carries limits of sequences) yields

$$f(x) = \lim_{n\to\infty} G_n(x), \qquad x \in (0,1].$$

(e) The one-step ratio of G_n mimics the functional equation. For any $z \geq 0$,

$$\begin{aligned}\frac{G_n(z+1)}{G_n(z)} &= \frac{n^{z+1}n!}{\prod_{k=1}^n(z+1+k)} \cdot \frac{\prod_{k=1}^n(z+k)}{n^z n!}\\ &= n\cdot\frac{z+1}{z+n+1} \;\longrightarrow\; z+1 \quad (n\to\infty),\end{aligned}$$

since $n/(n+z+1) \to 1$. Consequently, if $G_n(z) \to f(z)$ then by the Algebraic Limit Theorem (Theorem 2.3.3)

$$G_n(z+1) = \frac{G_n(z+1)}{G_n(z)}\cdot G_n(z) \;\longrightarrow\; (z+1)f(z) = f(z+1),$$

the last equality by (ii). Now let $x \geq 0$ be arbitrary. If $x = 0$ then $G_n(0) = n!/n! = 1 = f(0)$ for every n, so the formula holds trivially. If $x > 0$, write $x = m + y$ with $m \in \mathbf{N}\cup\{0\}$ and $y \in (0,1]$. Part (d) gives $G_n(y) \to f(y)$, and m applications of the displayed step give $G_n(y+m) \to f(y+m)$, i.e.

$$f(x) = \lim_{n\to\infty}\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)} \qquad \text{for all } x \geq 0.$$

Problem 8.4.22. (Recall that when $x!$ is extended to all of $\mathbf{R}$ via the functional equation $x! = x\,(x-1)!$ of Exercise 8.4.20(b), the extended factorial has a vertical asymptote at each of $x = -1,-2,-3,\ldots$, so the reciprocal function $1/x!$ is taken to be zero at $x = -1,-2,-3,\ldots$.)
(a) Where does

$$g(x) = \frac{x}{x!\,(-x)!}$$

equal zero? What other familiar function has the same set of roots?
(b) The function e^{-x^2} provides the raw material for the all-important Gaussian bell curve from probability, where it is known that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Use this fact (and some standard integration techniques) to evaluate $(1/2)!$.
(c) Now use (a) and (b) to conjecture a striking relationship between the factorial function and a well-known function from trigonometry.

Solution. (a) g vanishes exactly on the integers, which is precisely the root set of $\sin(\pi x)$.
Factor $g(x) = x\cdot\dfrac{1}{x!}\cdot\dfrac{1}{(-x)!}$ and run the cases:
(i) $x = 0$: the factor x vanishes, and $0! = 1$ leaves nothing indeterminate.
(ii) $x = n$ with $n \in \mathbf{N}$: $(-n)!$ sits on an asymptote, so $1/(-x)! = 0$.
(iii) $x = -n$ with $n \in \mathbf{N}$: symmetrically $1/x! = 0$.
(iv) $x \notin \mathbf{Z}$: neither x nor $-x$ is a negative integer, so $x!$ and $(-x)!$ are finite, and both are nonzero because the extended factorial is never zero (p. 280); also $x \neq 0$.
Hence

$$g(x) = 0 \iff x \in \mathbf{Z},$$

the same zero set as $\sin(\pi x)$.

(b) $(1/2)! = \dfrac{\sqrt{\pi}}{2}$.
From Definition 8.4.10, substituting $t = u^2$ ($dt = 2u\,du$) and then integrating by parts with $dv = ue^{-u^2}du$:

$$\begin{aligned}\left(\tfrac{1}{2}\right)! &= \int_0^\infty t^{1/2}e^{-t}\,dt = 2\int_0^\infty u^2e^{-u^2}\,du\\ &= 2\left(\left[-\tfrac{u}{2}e^{-u^2}\right]_0^\infty + \tfrac{1}{2}\int_0^\infty e^{-u^2}\,du\right)\\ &= \int_0^\infty e^{-u^2}\,du = \tfrac{1}{2}\int_{-\infty}^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2},\end{aligned}$$

the last two steps using evenness of e^{-u^2} and the given Gaussian integral.
(c) The conjecture is the reflection formula

$$x!\,(-x)! \;=\; \frac{\pi x}{\sin(\pi x)}, \qquad \text{equivalently} \qquad g(x) = \frac{\sin(\pi x)}{\pi}.$$

Both sides of the second identity are zero exactly on $\mathbf{Z}$ by (a), and the constant $1/\pi$ is calibrated by $x = 1/2$: the functional equation gives $(1/2)! = \frac{1}{2}(-1/2)!$, so $(-1/2)! = 2\left(\frac{1}{2}\right)! = \sqrt{\pi}$ by (b), whence

$$g\left(\tfrac{1}{2}\right) = \frac{1/2}{(\sqrt{\pi}/2)\sqrt{\pi}} = \frac{1}{\pi} = \frac{\sin(\pi/2)}{\pi}.$$

Problem 8.4.23. As a parting shot, use the value for $(1/2)!$ and the Gauss product formula in equation (9),

$$x! = \lim_{n\to\infty} \frac{n^x\,n!}{(x+1)(x+2)\cdots(x+n)},$$

to derive the famous product formula for π discovered by John Wallis in the 1650s:

$$\frac{\pi}{2} = \lim_{n\to\infty}\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

Solution. Put $x = 1/2$ in (9) and clear the halves. Abbreviate

$$E_n = 2\cdot 4\cdot 6\cdots(2n) = 2^n n!, \qquad O_n = 3\cdot 5\cdot 7\cdots(2n+1).$$

The denominator of (9) is $\prod_{k=1}^n\left(\frac{1}{2}+k\right) = 2^{-n}O_n$, so with $(1/2)! = \sqrt{\pi}/2$ from Exercise 8.4.22(b),

$$\begin{aligned}\frac{\sqrt{\pi}}{2} = \left(\tfrac{1}{2}\right)! = \lim_{n\to\infty}\frac{\sqrt{n}\,n!}{2^{-n}O_n} &= \lim_{n\to\infty}\sqrt{n}\,\frac{2^n n!}{O_n}\\ &= \lim_{n\to\infty}\sqrt{n}\,\frac{E_n}{O_n}.\end{aligned}$$

Squaring, legitimate by the Algebraic Limit Theorem 2.3.3(iii) since this limit exists,

$$\lim_{n\to\infty} n\,\frac{E_n^2}{O_n^2} = \frac{\pi}{4}.$$

The Wallis partial product has numerator $(2\cdot 4\cdots 2n)^2 = E_n^2$ and denominator $\big(1\cdot 3\cdots(2n-1)\big)\big(3\cdot 5\cdots(2n+1)\big)$, where the first factor is $O_n/(2n+1)$ and the second is O_n:

$$W_n = \prod_{k=1}^n \frac{2k\cdot 2k}{(2k-1)(2k+1)} = \frac{E_n^2}{\big(O_n/(2n+1)\big)\,O_n} = (2n+1)\,\frac{E_n^2}{O_n^2}.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} W_n &= \lim_{n\to\infty}\left(2+\frac{1}{n}\right)\cdot n\,\frac{E_n^2}{O_n^2}\\ &= 2\cdot\frac{\pi}{4} = \frac{\pi}{2},\end{aligned}$$

a product of two convergent sequences, again by Theorem 2.3.3(iii).

Problem 8.5.1. Recall d'Alembert's model for a vibrating string of length π fastened at both ends, where $u(x,t)$ is the displacement at position $x \in [0,\pi]$ and time $t \geq 0$:

$$(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

$$(2) \quad u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0 \quad \text{for all } t \geq 0,$$

$$(3) \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Verify that

$$u(x,t) = b_n \sin(nx)\cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbf{N}$ and $b_n \in \mathbf{R}$. What goes wrong if $n \notin \mathbf{N}$?
(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Solution. (a) Both second derivatives equal $-n^2 u$:

$$\frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx)\cos(nt),$$
$$\frac{\partial^2 u}{\partial t^2} = -n^2 b_n \sin(nx)\cos(nt),$$

so (1) holds. For (2), $u(0,t) = b_n \sin(0)\cos(nt) = 0$ and

$$u(\pi,t) = b_n \sin(n\pi)\cos(nt) = 0$$

because n is an integer. For (3),

$$\frac{\partial u}{\partial t}(x,t) = -n\, b_n \sin(nx)\sin(nt),$$

which vanishes at $t = 0$ since $\sin(0) = 0$.
If $n \notin \mathbf{N}$, only (2) breaks: the computations for (1) and (3) never used integrality, but $\sin(n\pi) = 0$ forces $n \in \mathbf{Z}$, so for non-integer n we get $u(\pi,t) = b_n \sin(n\pi)\cos(nt) \not\equiv 0$ and the string is not pinned at $x = \pi$. The integers $n \leq 0$ are excluded only as redundant: $n = 0$ gives $u \equiv 0$, and $-n$ repeats n with b_n replaced by $-b_n$.
(b) Each of (1), (2), (3) is a linear homogeneous condition, so for $u = \sum_{n=1}^{N} u_n$ with each u_n as in part (a),

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{N} \left(\frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_n}{\partial t^2} \right) = 0,$$
$$u(0,t) = \sum_{n=1}^{N} u_n(0,t) = 0,$$
$$u(\pi,t) = \sum_{n=1}^{N} u_n(\pi,t) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{N} \frac{\partial u_n}{\partial t}(x,0) = 0.$$

Finiteness is what legitimizes pulling the derivatives inside: it is the algebraic sum rule for derivatives applied $N-1$ times, with no convergence hypothesis needed.

Problem 8.5.2. Using trigonometric identities when necessary, verify the following integrals.
(a) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0.$$

(b) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi.$$

(c) For all $m, n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0.$$

Solution. (a) Antidifferentiate, using $\sin(\pm n\pi) = 0$ and the evenness of cosine:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(nx)\,dx &= \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi} = \frac{\sin(n\pi) - \sin(-n\pi)}{n} = 0,\\
\int_{-\pi}^{\pi} \sin(nx)\,dx &= \left.\frac{-\cos(nx)}{n}\right|_{-\pi}^{\pi} = \frac{-\cos(n\pi) + \cos(-n\pi)}{n} = 0.
\end{aligned}$$

(The second is immediate anyway: $\sin(nx)$ is odd and the interval is symmetric.)
(b) Use the half-angle identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, then apply part (a) with the integer $2n$:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(2nx)\,dx = \frac{1}{2}(2\pi) + 0 = \pi,\\
\int_{-\pi}^{\pi} \sin^2(nx)\,dx &= \frac{1}{2}(2\pi) - 0 = \pi.
\end{aligned}$$

(c) The product-to-sum identities turn every product into a sum of two terms of the type handled in (a). First,

$$\cos(mx)\sin(nx) = \tfrac{1}{2}\big[\sin((n+m)x) + \sin((n-m)x)\big].$$

Here $n + m \in \mathbf{N}$, so the first term integrates to 0 by (a); the second integrates to 0 by (a) if $n > m$, by oddness of sin if $n < m$, and trivially if $n = m$ (the integrand is $\sin(0) = 0$). Hence the integral is 0 for all m, n.
Next, for $m \neq n$,

$$\begin{aligned}
\cos(mx)\cos(nx) &= \tfrac{1}{2}\big[\cos((m-n)x) + \cos((m+n)x)\big],\\
\sin(mx)\sin(nx) &= \tfrac{1}{2}\big[\cos((m-n)x) - \cos((m+n)x)\big].
\end{aligned}$$

Since $m \neq n$, both $|m-n|$ and $m+n$ are positive integers, and cos is even, so each of the four integrals vanishes by (a). Therefore both displayed integrals equal 0.

8.11 Exercises 8.5.3–8.5.9

Problem 8.5.3. Assuming the representation

$$(6) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

derive the formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad \text{and} \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \tag{10}$$

for all $m \geq 1$.

Solution. Fix $m \geq 1$, multiply (6) by $\cos(mx)$, integrate over $[-\pi, \pi]$, and (as with a_0 in equation (9)) brazenly interchange the integral and the sum:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_0 &\int_{-\pi}^{\pi} \cos(mx)\,dx \\ &+ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx \\ &+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx.\end{aligned}$$

By Exercise 8.5.2(a) the first term is 0; by Exercise 8.5.2(c) every integral in the second sum with $n \neq m$ is 0 and every integral in the third sum is 0. The single survivor is the $n = m$ term of the second sum, which by Exercise 8.5.2(b) equals $a_m\pi$. Thus

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\pi,$$

which is the first formula in (10).
Multiplying (6) by $\sin(mx)$ instead and integrating gives, by the same three citations,

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = 0 + 0 + b_m \int_{-\pi}^{\pi} \sin^2(mx)\,dx = b_m\pi,$$

which is the second formula in (10).

Problem 8.5.4. Example 8.5.1 computes the Fourier series of the odd step function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ 0 & \text{if } x = 0 \text{ or } x = \pi \\ -1 & \text{if } -\pi < x < 0, \end{cases}$$

obtaining $a_n = 0$ for all $n \geq 0$ and $b_n = 4/(n\pi)$ for n odd, $b_n = 0$ for n even, so that

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin((2n+1)x).$$

(a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is not uniform on any interval containing 0.
(b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?
(c) Use graphs to collect some empirical evidence regarding the question of term-by-term differentiation in our two examples to this point. Is it possible to conclude convergence or divergence of either differentiated series by looking at the resulting coefficients? Theorem 6.4.3 is about the legitimacy of term-by-term differentiation. Can it be applied to either of these examples?

Solution. (a) Because f is discontinuous at 0 while every partial sum is continuous. Each

$$S_N(x) = \frac{4}{\pi}\sum_{n=0}^{N}\frac{\sin((2n+1)x)}{2n+1}$$

is a finite sum of continuous functions, hence continuous. If $S_N \to f$ uniformly on an interval I containing 0 (with more than one point), the Continuous Limit Theorem (Theorem 6.2.6) would force f to be continuous at 0 relative to I. But $f(0) = 0$ while $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$, so f fails to be continuous at 0 from whichever side I approaches it. Hence the convergence is not uniform on I.
(b) Since $g(x) = |x|$ is even, $g(x)\sin(nx)$ is odd and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)\,dx = 0 \quad \text{for all } n \geq 1.$$

The remaining coefficients, using evenness of $g(x)\cos(nx)$ and integration by parts:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\int_0^{\pi}x\,dx = \frac{\pi}{2},\\
a_n &= \frac{2}{\pi}\int_0^{\pi}x\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi}\\
&= \frac{2}{\pi}\cdot\frac{(-1)^n-1}{n^2} = \begin{cases} -4/(\pi n^2) & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

Hence

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{(2n+1)^2},$$

and the partial sums are visibly indistinguishable from the triangle wave already at $N = 3$ or 4, with no Gibbs overshoot near 0. Uniform convergence is read straight off the coefficients: since

$$\left|\frac{4}{\pi}\cdot\frac{\cos((2n+1)x)}{(2n+1)^2}\right| \leq \frac{4}{\pi(2n+1)^2} =: M_n \quad \text{and} \quad \sum M_n < \infty$$

(a convergent p-series, $p = 2$), the Weierstrass M-Test (Corollary 6.4.5) gives uniform convergence on all of $\mathbf{R}$ to some continuous limit.
(c) Differentiating Example 8.5.1 term by term produces

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\cos((2n+1)x),$$

whose partial sums on a graph oscillate with growing amplitude and settle nowhere. The coefficients decide the matter outright: at $x = 0$ every term equals $4/\pi$, so the terms do not tend to 0 and the series diverges by the nth term test.
Differentiating the series in (b) term by term produces

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{2n+1},$$

which is exactly the series of Example 8.5.1 – and the graphs show it converging to the step function $\operatorname{sgn}(x)$, which is indeed $g'(x)$ for $x \neq 0$. Here the coefficients alone are not enough to decide: $4/(\pi(2n+1)) \to 0$, but $\sum 1/(2n+1)$ diverges, so the M-Test is silent and no comparison of coefficients settles convergence.
Theorem 6.4.3 applies to neither example. Its hypothesis is that the differentiated series $\sum f_n'$ converge uniformly – not merely pointwise – on the interval. For Example 8.5.1 the differentiated series does not converge at all, and for $g = |x|$ the differentiated series is the one from Example 8.5.1, whose convergence is not uniform on any interval containing 0 by part (a).

Problem 8.5.5. This exercise is a step in the proof of Theorem 8.5.2 (the Riemann–Lebesgue Lemma), which asserts that if h is continuous on $(-\pi, \pi]$ then

$$\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx \to 0 \quad \text{and} \quad \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx \to 0$$

as $n \to \infty$. As always, h is mentally extended to be 2π-periodic, and the continuity hypothesis is intended to mean that this periodic extension is continuous on all of $\mathbf{R}$; in particular $\lim_{x\to -\pi^+} h(x) = h(\pi)$.
Explain why h is uniformly continuous on $\mathbf{R}$.

Solution. Periodicity reduces the claim to a compact interval. The extended h is continuous on the compact set $[-2\pi, 2\pi]$, so it is uniformly continuous there by Theorem 4.4.7 (Uniform Continuity on Compact Sets).
Given $\epsilon > 0$, choose $\delta_0 > 0$ witnessing this on $[-2\pi, 2\pi]$ and set $\delta = \min\{\delta_0, \pi\}$. Now let $x, y \in \mathbf{R}$ with $|x - y| < \delta$. Pick $k \in \mathbf{Z}$ with $x - 2k\pi \in (-\pi, \pi]$; then

$$|y - 2k\pi| \le |x - 2k\pi| + |x - y| < \pi + \pi = 2\pi,$$

so $x - 2k\pi$ and $y - 2k\pi$ both lie in $[-2\pi, 2\pi]$ and are still within δ_0 of each other. Since h is 2π-periodic,

$$|h(x) - h(y)| = |h(x - 2k\pi) - h(y - 2k\pi)| < \epsilon.$$

Thus h is uniformly continuous on $\mathbf{R}$.

Problem 8.5.6. Continuing the proof of Theorem 8.5.2 (the Riemann–Lebesgue Lemma): given $\epsilon > 0$, choose $\delta > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \epsilon/2$ (possible by Exercise 8.5.5). The period of $\sin(nx)$ is $2\pi/n$, so choose N large enough so that $\pi/n < \delta$ whenever $n \ge N$. Now consider a particular interval $[a, b]$ of length $2\pi/n$ over which $\sin(nx)$ moves through one complete oscillation.
Show that $\left|\int_a^b h(x)\sin(nx)\,dx\right| < \epsilon/n$, and use this fact to complete the proof.

Solution. Pair each half-oscillation against the next one. That $\sin(nx)$ moves through one complete oscillation on $[a, b]$ means na is an integer multiple of π; writing $c = a + \pi/n$ for the midpoint, $\sin(nx)$ has one sign on $[a, c]$ and the opposite sign on $[c, b]$. Substituting $x = s + \pi/n$ in the second half and using $\sin(ns + \pi) = -\sin(ns)$,

$$\begin{aligned} \int_a^b h(x)\sin(nx)\,dx &= \int_a^c h(s)\sin(ns)\,ds + \int_a^c h(s + \tfrac{\pi}{n})\sin(ns + \pi)\,ds \\ &= \int_a^c \big[h(s) - h(s + \tfrac{\pi}{n})\big]\sin(ns)\,ds. \end{aligned}$$

Now $n \ge N$ gives $\pi/n < \delta$, so the continuous function $s \mapsto |h(s) - h(s + \pi/n)|$ is bounded by $\epsilon/2$ on the compact interval $[a, c]$, and being continuous it attains a maximum $M < \epsilon/2$ there. Since $\sin(ns)$ has constant sign on $[a, c]$,

$$\int_a^c |\sin(ns)|\,ds = \frac{1}{n}\int_{na}^{na+\pi} |\sin u|\,du = \frac{2}{n},$$

and therefore

$$\left|\int_a^b h(x)\sin(nx)\,dx\right| \le M\int_a^c |\sin(ns)|\,ds \le \frac{2M}{n} < \frac{\epsilon}{n}.$$

To complete the proof, cut $(-\pi, \pi]$ into exactly n such intervals. Put

$$I_k = \left[-\pi + \frac{2k\pi}{n},\ -\pi + \frac{2(k+1)\pi}{n}\right], \qquad k = 0, 1, \ldots, n-1.$$

Each has length $2\pi/n$, and the left endpoint a_k satisfies $na_k = -n\pi + 2k\pi$, an integer multiple of π, so $\sin(nx)$ runs through one complete oscillation on I_k exactly as above. Hence for every $n \geq N$,

$$\left|\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx\right| \leq \sum_{k=0}^{n-1}\left|\int_{I_k} h(x)\sin(nx)\,dx\right| < n\cdot\frac{\epsilon}{n} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx \to 0$.
For the cosine statement, repeat the argument with the partition shifted by a quarter period so that the cut points are the zeros of $\cos(nx)$. Because $h(x)\cos(nx)$ is 2π-periodic (n is an integer), the integral over $(-\pi,\pi]$ equals the integral over $[\alpha, \alpha+2\pi]$ for $\alpha = -\pi + \pi/(2n)$, and $n\alpha = -n\pi + \pi/2$ is an odd multiple of $\pi/2$. Cutting $[\alpha, \alpha+2\pi]$ into n intervals of length $2\pi/n$ and using $\cos(ns+\pi) = -\cos(ns)$ in place of the sine identity gives the same bound ϵ/n on each piece, hence

$$\left|\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx\right| < \epsilon \qquad \text{for all } n \geq N.$$

Problem 8.5.7. This exercise completes the proof of Theorem 8.5.3: if f is continuous on $(-\pi,\pi]$ (extended 2π-periodically, so that $\lim_{x\to-\pi^+} f(x) = f(\pi)$) and S_N is the Nth partial sum of its Fourier series, then $S_N(x) \to f(x)$ at every $x \in (-\pi,\pi]$ where $f'(x)$ exists.
The proof has reached the identity

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\,du + \frac{1}{2\pi}\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\,du,$$

obtained by inserting the rewritten Dirichlet kernel

$$D_N(u) = \frac{\sin((N+1/2)u)}{2\sin(u/2)} = \frac{1}{2}\left[\frac{\sin(Nu)\cos(u/2)}{\sin(u/2)} + \cos(Nu)\right]$$

into equation (11), where

$$p_x(u) = \frac{\big(f(u+x) - f(x)\big)\cos(u/2)}{\sin(u/2)} \quad \text{and} \quad q_x(u) = f(u+x) - f(x).$$

(a) First, argue why the integral involving $q_x(u)$ tends to zero as $N \to \infty$.
(b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Solution. (a) Apply the Riemann–Lebesgue Lemma (Theorem 8.5.2) to $h = q_x$. Its hypothesis is met: with x fixed, $u \mapsto f(u+x)$ is the 2π-periodic extension of f composed with a translation, hence continuous on all of $\mathbf{R}$, and subtracting the constant $f(x)$ preserves that. Therefore

$$\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\,du \to 0 \qquad \text{as } N \to \infty,$$

and the second term of the displayed identity tends to 0.
(b) The same theorem applies to p_x, once we check it is continuous with continuous periodic extension – which is exactly where differentiability at x enters.
(i) On $(-\pi,\pi]\setminus\{0\}$: $\sin(u/2) = 0$ only when u is an integer multiple of 2π, so on $(-\pi,\pi]$ the denominator vanishes only at $u = 0$. Away from 0, p_x is a quotient of continuous functions with nonvanishing denominator, hence continuous.

(ii) At $u = 0$: factor out the difference quotient and use $\lim_{\theta \to 0} \sin(\theta)/\theta = 1$,

$$\begin{aligned} p_x(u) &= \frac{f(x+u) - f(x)}{u} \cdot \frac{u/2}{\sin(u/2)} \cdot 2\cos(u/2), \\ \lim_{u \to 0} p_x(u) &= f'(x) \cdot 1 \cdot 2 = 2f'(x), \end{aligned}$$

the first factor converging precisely because f is differentiable at x. Defining $p_x(0) = 2f'(x)$ therefore makes p_x continuous on $[-\pi, \pi]$, and altering an integrand at the single point $u = 0$ changes no integral.
(iii) At the ends: $\cos(\pm\pi/2) = 0$ while $\sin(\pm\pi/2) = \pm 1 \neq 0$, so

$$p_x(\pi) = 0 \quad \text{and} \quad \lim_{u \to -\pi^+} p_x(u) = \big(f(x-\pi) - f(x)\big) \cdot \frac{0}{-1} = 0,$$

so the 2π-periodic extension of p_x is continuous on all of $\mathbf{R}$ as well.
Thus p_x satisfies the hypothesis of Theorem 8.5.2, and

$$\int_{-\pi}^{\pi} p_x(u) \sin(Nu)\, du \to 0 \qquad \text{as } N \to \infty.$$

Combining with (a), both terms of the identity vanish in the limit, so $S_N(x) - f(x) \to 0$, i.e. $S_N(x) \to f(x)$ at every point x where $f'(x)$ exists.

Problem 8.5.8. Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converge to the same limit. Give an example to show that it is possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Solution. Split the average at the index past which (x_n) is already within $\epsilon/2$ of its limit.
Say $x_n \to L$ and let $\epsilon > 0$. Choose N so that $|x_k - L| < \epsilon/2$ for all $k > N$, and put $C = \sum_{k=1}^{N} |x_k - L|$, a constant once N is fixed. For $n > N$,

$$\begin{aligned} |y_n - L| &= \left| \frac{1}{n} \sum_{k=1}^{n} (x_k - L) \right| \le \frac{1}{n} \sum_{k=1}^{N} |x_k - L| + \frac{1}{n} \sum_{k=N+1}^{n} |x_k - L| \\ &< \frac{C}{n} + \frac{n-N}{n} \cdot \frac{\epsilon}{2} \le \frac{C}{n} + \frac{\epsilon}{2}. \end{aligned}$$

Since $C/n \to 0$, there is an $M \ge N$ with $C/n < \epsilon/2$ for all $n > M$, and then $|y_n - L| < \epsilon$. Hence $y_n \to L$.
For the converse failure take $x_n = (-1)^n$, which diverges, while

$$y_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^k = \begin{cases} 0, & n \text{ even}, \\ -1/n, & n \text{ odd}, \end{cases}$$

so $|y_n| \le 1/n \to 0$ and (y_n) converges to 0.

Problem 8.5.9. In the proof of Fejer's Theorem (Theorem 8.5.4) one needs the sine analogue of Fact 2 from the proof of Theorem 8.5.3, namely

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(N\theta) = \frac{\sin\left(\frac{N\theta}{2}\right) \sin\left((N+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)} \left[\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right]^2,$$

where D_n is the Dirichlet kernel of Fact 3 in the proof of Theorem 8.5.3,

$$D_n(\theta) = \begin{cases} \dfrac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}, & \theta \neq 2k\pi, \\ 1/2 + n, & \theta = 2k\pi. \end{cases}$$

Solution. The leading $1/2$ is exactly $D_0(\theta)$, so the claim is the closed form

$$\begin{aligned} \sum_{n=0}^{N} D_n(\theta) &= \frac{1}{2\sin(\theta/2)} \sum_{n=0}^{N} \sin\left(\left(n + \tfrac{1}{2}\right)\theta\right) \\ &= \frac{1}{2} \left[\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right]^2. \end{aligned}$$

Fix $\theta \neq 2k\pi$ and write $\psi = \theta/2$, so that the middle sum is $S = \sum_{n=0}^{N} \sin((2n+1)\psi)$, the odd-indexed part of a full sine sum. Subtracting the even part,

$$S = \sum_{k=1}^{2N+1} \sin(k\psi) \; - \; \sum_{k=1}^{N} \sin(k\,(2\psi)),$$

and the printed identity applied to each piece (first with angle ψ and $2N+1$ terms, then with angle 2ψ and N terms) gives

$$S = \sin((N+1)\psi) \left[\frac{\sin\left(\frac{(2N+1)\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} - \frac{\sin(N\psi)}{\sin(\psi)} \right].$$

The bracket collapses: by the sum-to-product formula and $\sin\psi = 2\sin(\psi/2)\cos(\psi/2)$,

$$\begin{aligned} \frac{\sin(N\psi) + \sin((N+1)\psi)}{\sin\psi} &= \frac{2\sin\left(\frac{(2N+1)\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)}{2\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)} \\ &= \frac{\sin\left(\frac{(2N+1)\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}, \end{aligned}$$

so the bracket equals $\sin((N+1)\psi)/\sin(\psi)$ and therefore

$$S = \frac{\sin^2((N+1)\psi)}{\sin(\psi)} = \frac{\sin^2\left((N+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}.$$

Dividing by $2\sin(\theta/2)$ and then by $N+1$ gives the asserted formula. At the excluded angles $\theta = 2k\pi$ both sides equal $(N+1)/2$, the right side read as its limiting value. (Check!)
Method (2): multiply term by term instead. Since $2\sin(\theta/2)\sin((n+1/2)\theta) = \cos(n\theta) - \cos((n+1)\theta)$, the sum telescopes,

$$2\sin(\theta/2)\, S = 1 - \cos((N+1)\theta) = 2\sin^2\left((N+1)\tfrac{\theta}{2}\right),$$

which is the same closed form for S.

8.12 Exercises 8.5.10–8.6.5

Problem 8.5.10. The expression in Exercise 8.5.9 is called the Fejer kernel and is denoted by

$$F_N(\theta) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(\theta) = \frac{1}{2(N+1)}\left[\frac{\sin\big((N+1)\frac{\theta}{2}\big)}{\sin\big(\frac{\theta}{2}\big)}\right]^2.$$

Let f be continuous on $(-\pi,\pi]$, extended 2π-periodically to $\mathbf{R}$, let $S_n(x)$ be the nth partial sum of its Fourier series, and let $\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x)$.
(a) Show that

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\, F_N(u)\, du.$$

(b) Graph the function $F_N(u)$ for several values of N. Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \to 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi,\pi]$).
(c) Prove that $\int_{-\pi}^{\pi} F_N(u)\, du = \pi$.
(d) To finish the proof of Fejer's Theorem, first choose a $\delta > 0$ so that

$$|u| < \delta \quad \text{implies} \quad |f(x+u) - f(x)| < \epsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x.

Solution. (a) Average the Dirichlet-kernel formula for S_n. The proof of Theorem 8.5.3 established $S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)D_n(u)\, du$ for every $n \geq 0$ (for $n = 0$ this reads $\frac{1}{2\pi}\int_{-\pi}^{\pi} f = a_0$, since $D_0 \equiv 1/2$). Averaging and exchanging the finite sum with the integral by linearity,

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)D_n(u)\, du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\left[\frac{1}{N+1}\sum_{n=0}^{N} D_n(u)\right] du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\, du,
\end{aligned}$$

the bracket being $F_N(u)$ by Exercise 8.5.9.
(b) F_N is a squared quotient, hence $F_N \geq 0$ everywhere – the decisive difference from D_N, which oscillates in sign with an envelope $1/(2|\sin(u/2)|)$ that does not decay in N. It spikes at $u = 0$, where $F_N(0) = (N+1)/2$, and vanishes at the points $u = 2k\pi/(N+1)$ lying in $[-\pi,\pi]$ with k a nonzero integer, so the spike narrows to width $O(1/N)$ as N grows while the side lobes are crushed toward zero.
For the uniform statement, fix $\delta > 0$ and let $\delta \leq |u| \leq \pi$. Then $|\sin(u/2)| \geq \sin(\delta/2) > 0$, and since $|\sin| \leq 1$ in the numerator,

$$0 \leq F_N(u) \leq \frac{1}{2(N+1)\sin^2(\delta/2)}.$$

The bound is independent of u and tends to 0, so $F_N \to 0$ uniformly on $\{u : \delta \leq |u| \leq \pi\}$.
(c) By Fact 3 in the proof of Theorem 8.5.3, $\int_{-\pi}^{\pi} D_n(u)\, du = \pi$ for each n, so

$$\begin{aligned}
\int_{-\pi}^{\pi} F_N(u)\, du &= \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(u)\, du \\
&= \frac{1}{N+1}(N+1)\pi = \pi.
\end{aligned}$$

(d) Multiplying (c) by $f(x)/\pi$ gives $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F_N(u)\,du$, so subtracting from (a) produces the single integral

$$\sigma_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(u+x) - f(x)\big)F_N(u)\,du.$$

The periodic extension of f is continuous on all of $\mathbf{R}$ (this is where the hypothesis is used; it requires $\lim_{x\to-\pi^+} f(x) = f(\pi)$, which is implicit in calling f continuous on the circle $(-\pi,\pi]$), so on the compact set $[-2\pi, 2\pi]$ it is bounded by Theorem 4.4.2, say $|f| \le M$, and uniformly continuous by Theorem 4.4.7; periodicity upgrades both to all of $\mathbf{R}$. Given $\epsilon > 0$, uniform continuity supplies an $\eta > 0$ with $|f(x+u) - f(x)| < \epsilon$ whenever $|u| < \eta$, one η for every x at once; put $\delta = \eta/2$ so the bound holds on the closed set $|u| \le \delta$. Split the integral there.
(i) On $|u| \le \delta$, use $F_N \ge 0$ together with (c):

$$\frac{1}{\pi}\int_{|u|\le\delta} |f(u+x) - f(x)|\,F_N(u)\,du \le \frac{\epsilon}{\pi}\int_{-\pi}^{\pi} F_N(u)\,du = \epsilon.$$

(ii) On $\delta \le |u| \le \pi$, use $|f(u+x) - f(x)| \le 2M$ and the bound from (b):

$$\begin{aligned}\frac{1}{\pi}\int_{\delta\le|u|\le\pi} 2M\,F_N(u)\,du &\le \frac{2M}{\pi}\cdot 2\pi \cdot \frac{1}{2(N+1)\sin^2(\delta/2)}\\ &= \frac{2M}{(N+1)\sin^2(\delta/2)}.\end{aligned}$$

Choose N_0 with $2M/\big((N+1)\sin^2(\delta/2)\big) < \epsilon$ for all $N \ge N_0$. Then $|\sigma_N(x) - f(x)| < 2\epsilon$ for every $N \ge N_0$ and every x: neither δ (from uniform continuity) nor N_0 (from the bound in (ii), which contains no x) depends on x. Hence $\sigma_N \to f$ uniformly, which is Theorem 8.5.4.

Problem 8.5.11. This exercise completes the derivation of the Weierstrass Approximation Theorem (Theorem 6.7.1: if $f : [a,b] \to \mathbf{R}$ is continuous and $\epsilon > 0$, then there is a polynomial p with $|f(x) - p(x)| < \epsilon$ for all $x \in [a,b]$) from Fejer's Theorem.
(a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a,b]$ is $[0,\pi]$.
(b) Show how the case for an arbitrary interval $[a,b]$ follows from this one.

Solution. (a) Reflect f evenly so that Fejer applies, then truncate the Taylor series of the finitely many trigonometric terms produced.
Let $f : [0,\pi] \to \mathbf{R}$ be continuous and $\epsilon > 0$. Define g on $(-\pi,\pi]$ by $g(x) = f(|x|)$. Then g is continuous, and $\lim_{x\to-\pi^+} g(x) = f(\pi) = g(\pi)$, so its 2π-periodic extension is continuous and Theorem 8.5.4 applies to g: its Fejer means σ_N converge to g uniformly, so fix N with

$$|\sigma_N(x) - g(x)| < \epsilon/2 \qquad \text{for all } x \in [-\pi,\pi].$$

Now σ_N, being an average of partial sums of a Fourier series, is a trigonometric polynomial,

$$\sigma_N(x) = c_0 + \sum_{n=1}^{N}\big(c_n\cos(nx) + d_n\sin(nx)\big),$$

with finitely many constants. Put $K = 1 + \sum_{n=1}^{N}(|c_n| + |d_n|)$. For each $n \le N$ the Taylor series

$$\begin{aligned}\cos(nx) &= \sum_{k=0}^{\infty}\frac{(-1)^k(nx)^{2k}}{(2k)!},\\ \sin(nx) &= \sum_{k=0}^{\infty}\frac{(-1)^k(nx)^{2k+1}}{(2k+1)!}\end{aligned}$$

converges uniformly on the compact set $[0,\pi]$, so we may choose partial sums P_n and Q_n – honest polynomials in x – with

$$\sup_{x\in[0,\pi]} |\cos(nx) - P_n(x)| < \frac{\epsilon}{2K},$$
$$\sup_{x\in[0,\pi]} |\sin(nx) - Q_n(x)| < \frac{\epsilon}{2K}.$$

Set $p(x) = c_0 + \sum_{n=1}^{N}\big(c_nP_n(x) + d_nQ_n(x)\big)$, a polynomial. For $x \in [0,\pi]$,

$$|\sigma_N(x) - p(x)| \le \sum_{n=1}^{N}\Big(|c_n|\tfrac{\epsilon}{2K} + |d_n|\tfrac{\epsilon}{2K}\Big) \le \frac{\epsilon}{2K}\,(K-1) < \frac{\epsilon}{2}.$$

Since $g(x) = f(x)$ on $[0,\pi]$, the triangle inequality gives $|f(x)-p(x)| < \epsilon$ there, which is WAT on $[0,\pi]$.
(b) Precompose with the affine bijection. Assume $a < b$ (if $a = b$ take the constant polynomial $p \equiv h(a)$) and let $h : [a,b] \to \mathbf{R}$ be continuous. The map

$$\varphi(t) = a + \frac{b-a}{\pi}\,t$$

is a continuous bijection of $[0,\pi]$ onto $[a,b]$ with continuous inverse $\varphi^{-1}(x) = \pi(x-a)/(b-a)$. Then $f = h \circ \varphi$ is continuous on $[0,\pi]$, so part (a) supplies a polynomial p with $|f(t) - p(t)| < \epsilon$ for all $t \in [0,\pi]$. Define

$$q(x) = p\left(\frac{\pi(x-a)}{b-a}\right),$$

which is again a polynomial in x, being the composition of a polynomial with a degree-one map. Given $x \in [a,b]$, set $t = \varphi^{-1}(x) \in [0,\pi]$; then $h(x) = f(t)$ and $q(x) = p(t)$, so

$$|h(x) - q(x)| = |f(t) - p(t)| < \epsilon.$$

Problem 8.6.1. Recall Definition 8.6.2: a subset A of the rational numbers is called a *cut* if it possesses the following three properties:
(c1) $A \neq \emptyset$ and $A \neq \mathbf{Q}$.
(c2) If $r \in A$, then A also contains every rational $q < r$.
(c3) A does not have a maximum; that is, if $r \in A$, then there exists $s \in A$ with $r < s$.
(a) Fix $r \in \mathbf{Q}$. Show that the set $C_r = \{t \in \mathbf{Q} : t < r\}$ is a cut.
The temptation to think of all cuts as being of this form should be avoided. Which of the following subsets of $\mathbf{Q}$ are cuts?
(b) $S = \{t \in \mathbf{Q} : t \le 2\}$
(c) $T = \{t \in \mathbf{Q} : t^2 < 2 \text{ or } t < 0\}$
(d) $U = \{t \in \mathbf{Q} : t^2 \le 2 \text{ or } t < 0\}$

Solution. (a) All three properties are one line each. (c1): $r - 1 \in C_r$ and $r \notin C_r$. (c2): if $t \in C_r$ and $q < t$, then $q < t < r$. (c3): if $t \in C_r$, then $s = (t + r)/2$ is rational and

$$t < \frac{t+r}{2} < r,$$

so $s \in C_r$ and $t < s$.
(b) Not a cut: $2 \in S$ is a maximum, so (c3) fails. (Properties (c1) and (c2) do hold.)
(c) T is a cut.
(c1): $0 \in T$ and $2 \notin T$.
(c2): let $t \in T$ and $q < t$. If $q < 0$ then $q \in T$. If $q \ge 0$, then $t > q \ge 0$, so the clause $t < 0$ fails for t and therefore $t^2 < 2$; since $0 \le q < t$ we get $q^2 < t^2 < 2$, so $q \in T$.

(c3): let $t \in T$. If $t < 0$, then $0 \in T$ and $t < 0$. If $t \geq 0$, so that $t^2 < 2$, set

$$s = t + \frac{2 - t^2}{t + 2} = \frac{2t + 2}{t + 2} \in \mathbf{Q}.$$

Then $s - t = (2 - t^2)/(t + 2) > 0$, and

$$\begin{aligned} 2 - s^2 &= \frac{2(t+2)^2 - 4(t+1)^2}{(t+2)^2} \\ &= \frac{4 - 2t^2}{(t+2)^2} > 0, \end{aligned}$$

so $s \in T$ with $t < s$.
(d) U is a cut, because $U = T$. Indeed $t^2 \leq 2$ and $t^2 < 2$ are the same condition on a rational t: equality $t^2 = 2$ would exhibit a rational square root of 2, which does not exist (Theorem 1.1.1). So (d) follows from (c).

Problem 8.6.2. Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Solution. Contrapose against (c2). The ordering of $\mathbf{Q}$ leaves exactly three possibilities for r and s:

$$\text{(i) } s = r, \quad \text{(ii) } s < r, \quad \text{(iii) } r < s.$$

(i) forces $s = r \in A$, contradicting $s \notin A$.
(ii) forces $s \in A$ by property (c2) of Definition 8.6.2 applied to $r \in A$, again contradicting $s \notin A$.
Hence (iii): $r < s$.

Problem 8.6.3. Recall Definition 8.6.4: a set F is a *field* if there exist two operations — addition $(x + y)$ and multiplication (xy) — that satisfy the following list of conditions:
(f1) (commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.
(f2) (associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.
(f3) (identities exist) There exist two special elements 0 and 1 with $0 \neq 1$ such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.
(f4) (inverses exist) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$. If $x \neq 0$, there exists an element x^{-1} such that $xx^{-1} = 1$.
(f5) (distributive property) $x(y + z) = xy + xz$ for all $x, y, z \in F$.
Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$, respectively.

Solution. Only $\mathbf{Q}$ is a field; $\mathbf{Z}$ misses exactly the multiplicative half of (f4), and $\mathbf{N} = \{1, 2, 3, \dots\}$ misses (f3) and (f4) entirely.

property	$\mathbf{N}$	$\mathbf{Z}$	$\mathbf{Q}$
(f1)	yes	yes	yes
(f2)	yes	yes	yes
(f3)	no	yes	yes
(f4)	no	additive only	yes
(f5)	yes	yes	yes

(f1), (f2), (f5) hold in all three, being inherited from the arithmetic of $\mathbf{Q}$ (each of $\mathbf{N}$, $\mathbf{Z}$ is closed under $+$ and $\cdot$, and the identities are equations among the same rationals).
(f3): $\mathbf{N}$ contains the multiplicative identity 1 but not the additive identity 0, so (f3) fails. $\mathbf{Z}$ and $\mathbf{Q}$ contain both, with $0 \neq 1$.
(f4): in $\mathbf{N}$ neither half holds, since $1 + x = 0$ and $2x = 1$ have no solutions $x \in \mathbf{N}$. In $\mathbf{Z}$ additive inverses exist ($-n \in \mathbf{Z}$), but $2x = 1$ has no integer solution, so multiplicative inverses fail. In $\mathbf{Q}$, $-p/q$ and q/p (for $p \neq 0$) supply both.

Problem 8.6.4. The real numbers $\mathbf{R}$ are defined (Definition 8.6.3) to be the set of all cuts in $\mathbf{Q}$. For $A, B \in \mathbf{R}$, define

$$A \leq B \quad \text{to mean} \quad A \subseteq B.$$

Show that this defines an ordering on $\mathbf{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5, namely:
(o1) For arbitrary $x, y \in F$, at least one of the statements $x \leq y$ or $y \leq x$ is true.
(o2) If $x \leq y$ and $y \leq x$, then $x = y$.
(o3) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Solution. (o2) and (o3) are the antisymmetry and transitivity of $\subseteq$: $A \subseteq B$ and $B \subseteq A$ give $A = B$ by definition of set equality, and $A \subseteq B \subseteq C$ gives $A \subseteq C$.
(o1) is the only place cut-hood is used. Suppose $A \not\subseteq B$; we show $B \subseteq A$. Pick $a \in A$ with $a \notin B$ and let $b \in B$ be arbitrary. Applying Exercise 8.6.2 to the cut B, with $b \in B$ and $a \notin B$, gives $b < a$; since $a \in A$ and A is a cut, property (c2) then gives $b \in A$. As $b \in B$ was arbitrary, $B \subseteq A$, i.e. $B \leq A$.

Problem 8.6.5. Given A and B in $\mathbf{R}$, define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

(The text verifies property (c2) of Definition 8.6.2 for $A+B$: if $s < a+b$ then $s - b < a$, so $s - b \in A$ and $s = (s-b) + b \in A + B$.)
(a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.
(b) Check that addition in $\mathbf{R}$ is commutative (f1) and associative (f2).
(c) Show that property (o4) holds, i.e. that $B \leq C$ implies $A + B \leq A + C$.
(d) Show that the cut

$$O = \{p \in \mathbf{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions: $A + O \subseteq A$ and $A \subseteq A + O$.)

Solution. (a) (c1): A and B are nonempty, so $A + B \neq \emptyset$. For $A + B \neq \mathbf{Q}$, choose $x \notin A$ and $y \notin B$ (possible by (c1) for A and B) and claim $x + y \notin A + B$: if $x + y = a + b$ with $a \in A$, $b \in B$, then $a < x$ and $b < y$ by Exercise 8.6.2, whence $a + b < x + y$, a contradiction.
(c3): given $a + b \in A + B$, property (c3) for A supplies $a' \in A$ with $a < a'$, and then

$$a + b < a' + b \in A + B.$$

With (c2) proved in the text, $A + B$ is a cut.
(b) Both are inherited from $\mathbf{Q}$:

$$\begin{aligned} A + B &= \{a + b\} = \{b + a\} = B + A, \\ (A + B) + C &= \{(a + b) + c\} \\ &= \{a + (b + c)\} = A + (B + C), \end{aligned}$$

the middle equality in each line being (f1), resp. (f2), for $\mathbf{Q}$. (For associativity, note that every element of $(A + B) + C$ has the displayed form $(a + b) + c$ with $a \in A$, $b \in B$, $c \in C$, and conversely.)
(c) If $B \leq C$, i.e. $B \subseteq C$, then every $a + b$ with $a \in A$, $b \in B$ is also of the form $a + c$ with $c = b \in C$. Hence $A + B \subseteq A + C$, i.e. $A + B \leq A + C$.
(d) O is a cut (it is C_0 in the notation of Exercise 8.6.1(a)). Two inclusions:
$A + O \subseteq A$: for $a \in A$ and $p < 0$ we have $a + p < a$, so $a + p \in A$ by (c2).
$A \subseteq A + O$: given $a \in A$, property (c3) supplies $a' \in A$ with $a < a'$; then

$$a = a' + (a - a'), \qquad a - a' < 0,$$

exhibits a as an element of $A + O$.

8.13 Exercises 8.6.6–8.6.9

Problem 8.6.6. Given $A \in \mathbf{R}$, define

$$-A = \{r \in \mathbf{Q} : \text{there exists } t \notin A \text{ with } t < -r\}.$$

(Conceptually $-A$ consists of all rationals less than $-\sup A$, but suprema are not yet available.)
(a) Prove that $-A$ defines a cut.
(b) What goes wrong if we set $-A = \{r \in \mathbf{Q} : -r \notin A\}$?
(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in Definition 8.6.4.

Solution. (a) (c1): $A \neq \mathbf{Q}$, so choose $u \notin A$; then $u + 1 \notin A$ as well (otherwise (c2) would put $u \in A$). The rational $r = -(u+1)$ lies in $-A$, with witness $t = u \notin A$ satisfying $t = u < u+1 = -r$. So $-A \neq \emptyset$. For $-A \neq \mathbf{Q}$, choose $a \in A$ and note $-a \notin -A$: a witness would be some $t \notin A$ with $t < -(-a) = a$, and (c2) would force $t \in A$.
(c2): if $r \in -A$ with witness $t \notin A$, $t < -r$, and $q < r$, then $t < -r < -q$, so the same t witnesses $q \in -A$.
(c3): with r, t as above, put $s = \big(t + (-r)\big)/2$, so that $t < s < -r$. Then $r' = -s$ satisfies $r < r'$, and $t < s = -r'$ shows $r' \in -A$.
(b) It can fail (c3). Take $A = O = \{p \in \mathbf{Q} : p < 0\}$. Then $-r \notin O$ exactly when $-r \geq 0$, so the proposed set is $\{r \in \mathbf{Q} : r \leq 0\}$, which has maximum 0. The same endpoint survives for every $A = C_q$, where the naive set is $\{r : r \leq -q\}$.
(c) Let $a \in A$ and $r \in -A$, with witness $t \notin A$, $t < -r$. Since $a \in A$ and $t \notin A$, Exercise 8.6.2 gives $a < t$. Hence

$$a < t < -r \quad \Longrightarrow \quad a + r < 0,$$

so $a + r \in O$ and $A + (-A) \subseteq O$.
For $O \subseteq A + (-A)$, fix $p < 0$ and set $\varepsilon = -p/2 > 0$. Fix $a_1 \in A$ and $u \notin A$. There is $n \in \mathbf{N}$ with $a_1 + n\varepsilon > u$ — this is Archimedean, but inside $\mathbf{Q}$ it needs no completeness: writing $(u - a_1)/\varepsilon = m/k$ with $m \in \mathbf{Z}$, $k \in \mathbf{N}$, the choice $n = |m| + 1 > m/k$ works. Then $a_1 + n\varepsilon \notin A$ (otherwise (c2) would put $u \in A$). So the set of integers $n \geq 0$ with $a_1 + n\varepsilon \notin A$ is nonempty; let N be its least element. Since $a_1 \in A$ we have $N \geq 1$, and by minimality

$$a := a_1 + (N-1)\varepsilon \in A, \qquad t := a_1 + N\varepsilon \notin A.$$

Put $r = p - a$. Then

$$-r = a - p = a + 2\varepsilon = a_1 + (N+1)\varepsilon > t,$$

so t witnesses $r \in -A$, and $a + r = p$. Hence $p \in A + (-A)$.
Therefore $A + (-A) = O$, which is (f4) for addition.

Problem 8.6.7. Given $A \geq O$ and $B \geq O$ in $\mathbf{R}$, define the product

$$\begin{aligned} AB = \{ab : a \in A,\ b \in B \text{ with } a, b \geq 0\} \\ \cup \{q \in \mathbf{Q} : q < 0\}. \end{aligned}$$

(a) Show that AB is a cut and that property (o5) holds, i.e. that $A \geq O$ and $B \geq O$ imply $AB \geq O$.
(b) Propose a good candidate for the multiplicative identity (1) on $\mathbf{R}$ and show that this works for all cuts $A \geq O$.
(c) Show the distributive property (f5) holds for non-negative cuts.

Solution. (a) Property (o5) is free: AB contains every negative rational by construction, i.e. $O \subseteq AB$, which is exactly $AB \geq O$. For cut-hood, note first the dichotomy used throughout: a cut $A \geq O$ (that is, $O \subseteq A$) either equals O, in which case A has no non-negative element and the product part of AB

is empty, so $OB = O$ for every $B \geq O$; or contains some $a_0 \geq 0$, and then (c3) produces $a^* \in A$ with $a^* > a_0 \geq 0$ while (c2) gives $0 \in A$. Call A *strictly positive* in the second case.
(c1): $AB \supseteq \{q < 0\} \neq \emptyset$. For $AB \neq \mathbf{Q}$: if $A = O$ or $B = O$ then $AB = O \neq \mathbf{Q}$. Otherwise both are strictly positive; choose $x \notin A$, $y \notin B$. By Exercise 8.6.2, x exceeds the positive element $a^* \in A$ and y exceeds a positive element of B, so $x, y > 0$. Then $xy \notin AB$: it is not negative, and if $xy = ab$ with $a \in A$, $b \in B$, $a, b \geq 0$, then $a < x$ and $b < y$ (Exercise 8.6.2), so

$$ab \leq ay < xy,$$

a contradiction.
(c2): let $z \in AB$ and $q < z$. If $q < 0$ then $q \in AB$. If $q \geq 0$ then $z > 0$, so $z = ab$ with $a, b \geq 0$, and $ab > 0$ forces $a > 0$, $b > 0$. Put $b' = q/a$, so $0 \leq b' < b$; then $b' \in B$ by (c2) for B, and $q = ab' \in AB$.
(c3): let $z \in AB$. If $z < 0$ then $z < z/2 < 0$ and $z/2 \in AB$. If $z = ab \geq 0$ with $a, b \geq 0$, use (c3) for A and B to get $a' > a$ in A and $b' > b$ in B; then $a', b' > 0$ and

$$a'b' > ab' \geq ab = z,$$

with $a'b' \in AB$.
(b) Take $I = C_1 = \{q \in \mathbf{Q} : q < 1\}$, a cut by Exercise 8.6.1(a), and note $I \geq O$. Two inclusions, for any $A \geq O$.
$AI \subseteq A$: negative rationals lie in A because $O \subseteq A$. For a product ai with $a \in A$, $i \in I$, $a, i \geq 0$, we have $0 \leq i < 1$; if $a = 0$ then $ai = 0 = a \in A$, and if $a > 0$ then $ai < a$, so $ai \in A$ by (c2).
$A \subseteq AI$: let $a \in A$. If $a < 0$, then $a \in AI$. If $a \geq 0$, property (c3) gives $a' \in A$ with $a' > a \geq 0$, so $a' > 0$; then $i = a/a'$ satisfies $0 \leq i < 1$, hence $i \in I$, and $a = a'i \in AI$.
(c) Claim: $A(B + C) = AB + AC$ for $A, B, C \geq O$. (Note $B + C \geq O$, since (o4) gives $B + C \geq B + O = B \geq O$, so both sides are defined.)
Degenerate cases first. If $A = O$, both sides equal O: the product parts are empty, and $O + O = O$ by Exercise 8.6.5(d). If $B = O$, then $AB = O$ and $B + C = C$, so the left side is AC and the right side is $O + AC = AC$. Symmetrically if $C = O$.
Assume now A, B, C are all strictly positive, so $0 \in A$, $0 \in B$, $0 \in C$. Then $0 = 0 \cdot 0 \in AB$, likewise $0 \in AC$, and hence $0 = 0 + 0 \in AB + AC$.
$A(B + C) \subseteq AB + AC$: let $z \in A(B + C)$. If $z < 0$, then $z < 0 \in AB + AC$, so $z \in AB + AC$ by (c2). If $z \geq 0$, write $z = ad$ with $a \in A$, $d \in B + C$, $a, d \geq 0$, and $d = b + c$ with $b \in B$, $c \in C$. Then $z = ab + ac$, and $ab \in AB$ in every case: if $b \geq 0$ this is the definition; if $b < 0$ and $a > 0$ then $ab < 0$; if $b < 0$ and $a = 0$ then $ab = 0 \in AB$. Likewise $ac \in AC$. Hence $z \in AB + AC$.
$AB + AC \subseteq A(B + C)$: let $u \in AB$ and $v \in AC$. Since $0 \in AB$ and $0 \in AC$, the rationals $u' = \max(u, 0)$ and $v' = \max(v, 0)$ satisfy $u' \in AB$, $v' \in AC$, and $u + v \leq u' + v'$. Being non-negative members, they are products: $u' = a_1 b_1$ and $v' = a_2 c_1$ with $a_1, a_2 \in A$, $b_1 \in B$, $c_1 \in C$ all ≥ 0 (for $u' = 0$ take $a_1 = b_1 = 0$, legitimate since $0 \in A$ and $0 \in B$). With $a = \max(a_1, a_2) \in A$,

$$\begin{aligned} u + v &\leq a_1 b_1 + a_2 c_1 \\ &\leq ab_1 + ac_1 = a(b_1 + c_1), \end{aligned}$$

and $a \geq 0$, $b_1 + c_1 \in B + C$ with $b_1 + c_1 \geq 0$, so $a(b_1 + c_1) \in A(B + C)$. If the inequality is strict, (c2) puts $u + v \in A(B + C)$; if it is equality, $u + v$ is that element. Either way $u + v \in A(B + C)$.

Problem 8.6.8. Let $\mathcal{A} \subseteq \mathbf{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.
(a) First, prove that $S \in \mathbf{R}$ by showing that it is a cut.
(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Solution. The union is the supremum: $S = \bigcup_{A \in \mathcal{A}} A$.
(a) Fix an upper bound $B \in \mathbf{R}$ for $\mathcal{A}$ (Definition 8.6.6), so $A \subseteq B$ for every $A \in \mathcal{A}$. Properties (c1)-(c3) of Definition 8.6.2:
(c1) Choose any $A_0 \in \mathcal{A}$, possible since $\mathcal{A} \neq \emptyset$. Then $A_0 \neq \emptyset$ by (c1) for the cut A_0, and $A_0 \subseteq S$, so

$S \neq \emptyset$. For the other half,

$$S = \bigcup_{A \in \mathcal{A}} A \subseteq B,$$

and B is a proper subset of $\mathbf{Q}$ by (c1) for the cut B, so $S \neq \mathbf{Q}$.
(c2) Let $r \in S$ and let $q \in \mathbf{Q}$ satisfy $q < r$. Then $r \in A$ for some $A \in \mathcal{A}$, and (c2) for A gives $q \in A \subseteq S$.
(c3) Let $r \in S$, say $r \in A$ with $A \in \mathcal{A}$. By (c3) for A there is $s \in A$ with $r < s$, and $s \in A \subseteq S$. So S has no maximum.
So S is a cut, i.e. $S \in \mathbf{R}$.
(b) The two criteria of Definition 8.6.6, with $\leq$ read as $\subseteq$.
(i) For each $A \in \mathcal{A}$, $A \subseteq \bigcup_{A' \in \mathcal{A}} A' = S$, so $A \leq S$.
(ii) If B is any upper bound for $\mathcal{A}$, then $A \subseteq B$ for every $A \in \mathcal{A}$, so

$$S = \bigcup_{A \in \mathcal{A}} A \subseteq B,$$

that is, $S \leq B$.

Problem 8.6.9. Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbf{Q} : t < r\}$$

where $r \in \mathbf{Q}$. (See Exercise 8.6.1.)
(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbf{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.
(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbf{Q}$.

Solution. The map $r \mapsto C_r$ is the promised embedding: it is additive, multiplicative on the non-negative cuts, and order-preserving in both directions. Each C_r is a cut by Exercise 8.6.1(a).
(a) *Addition.* By the definition of addition in $\mathbf{R}$, $C_r + C_s = \{a + b : a \in C_r,\ b \in C_s\}$.
($\subseteq$) If $a < r$ and $b < s$ then $a + b < r + s$, so $a + b \in C_{r+s}$.
($\supseteq$) Let $t \in C_{r+s}$ and put $\varepsilon = (r + s - t)/2 > 0$, a rational. Set

$$a = r - \varepsilon, \qquad b = t - a = (t - r) + \varepsilon.$$

Then $a < r$, and

$$\begin{aligned} b < s &\iff (t - r) + \varepsilon < s \\ &\iff \varepsilon < r + s - t = 2\varepsilon, \end{aligned}$$

which holds. So $t = a + b \in C_r + C_s$.
Multiplication, $r, s \geq 0$. By (b), $C_r, C_s \geq C_0 = O$, so the non-negative product definition applies:

$$\begin{aligned} C_r C_s &= \{ab : a \in C_r,\ b \in C_s,\ a, b \geq 0\} \\ &\quad \cup\ \{q \in \mathbf{Q} : q < 0\}. \end{aligned}$$

(i) $r = 0$ or $s = 0$. Then one factor is C_0, which contains no non-negative rational, so the first set is empty and $C_r C_s = \{q < 0\} = C_0 = C_{rs}$.
(ii) $r, s > 0$. For $\subseteq$: every $q < 0$ lies in C_{rs} because $rs > 0$; and if $0 \leq a < r$, $0 \leq b < s$, then

$$ab \leq as < rs,$$

using $b \leq s$ with $a \geq 0$, then $a < r$ with $s > 0$. So $ab \in C_{rs}$.
For $\supseteq$: let $t < rs$. If $t < 0$ then $t \in C_r C_s$ outright. If $t \geq 0$, then $t/s < r$, so take rational midpoints

$$t/s < a < r, \qquad t/a < b < s.$$

Here $a > t/s \geq 0$ makes $a > 0$, so $as > t$ and the second choice is legitimate; and $b > t/a \geq 0$. Thus $a \in C_r$, $b \in C_s$, $a, b \geq 0$, and

$$ab > a \cdot (t/a) = t.$$

Since $C_r C_s$ is a cut (Exercise 8.6.7(a)), property (c2) gives $t \in C_r C_s$. Therefore $C_r C_s = C_{rs}$.
(b) The ordering on $\mathbf{R}$ is $C_r \leq C_s$ if and only if $C_r \subseteq C_s$.
($\Leftarrow$) If $r \leq s$ and $t < r$, then $t < s$; so $C_r \subseteq C_s$.
($\Rightarrow$) Contrapositive: if $s < r$ then $s \in C_r$ while $s \notin C_s$, so $C_r \nsubseteq C_s$.